Summary

1 Dirichlet Beta Generating Functions

sech x , sec x and csc x can be expanded to Fourier series and Taylor series. And if the termwise higher order integration of these is carried out, Dirichlet Beta at a natural number are obtained.

Where, these are automorphisms which are expressed by lower betas. However, in this chapter, we stop those so far. The work that obtain the non-automorphism formulas by removing lower betas from these is done in the next chapter **" 2 Formulas for Dirichlet Beta " .**

In this chapter, we obtain the following polynomials from the beta generating functions of each family of sech, sec and csc . Where, Dirichlet Beta and Dirichlet Lambda are as follows.

$$\beta(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^x} \quad , \quad \lambda(x) = \sum_{r=1}^{\infty} \frac{1}{(2r-1)^x}$$

Bernoulli numbers and Euler numbers are as follows.

$$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30, \cdots$$
$$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$$

Harmonic number is $H_s = \sum_{t=1}^{s} 1/t = \psi(1+s)+\gamma$

$$\beta(n) = \sum_{r=0}^{\infty} \frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^n} - \frac{(-1)^n}{2} \sum_{r=0}^{\infty} \frac{E_{2r} x^{2r+n}}{(2r+n)!} - \sum_{s=1}^{n-1} \frac{(-1)^s x^s}{s!} \beta(n-s)$$

$$\sum_{r=0}^{\infty} \frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^{2n+1}} - \frac{(-1)^n}{2} \sum_{r=0}^{\infty} \frac{|E_{2r}| x^{2n+1+2r}}{(2n+1+2r)!} = \sum_{s=0}^{n-1} \frac{(-1)^s x^{2s+1}}{(2s+1)!} \beta(2n-2s)$$

$$\sum_{r=0}^{\infty} \frac{(-1)^r \cos\{(2r+1)x\}}{(2r+1)^{2n}} - \frac{(-1)^n}{2} \sum_{r=0}^{\infty} \frac{|E_{2r}| x^{2n+2r}}{(2n+2r)!} = \sum_{s=0}^{n-1} \frac{(-1)^s x^{2s}}{(2s)!} \beta(2n-2s)$$

$$\sum_{r=0}^{\infty} \frac{(-1)^r \cos\{(2r+1)x\}}{(2r+1)^{2n+1}} = \sum_{s=0}^{n} \frac{(-1)^s x^{2s}}{(2s)!} \beta(2n+1-2s)$$

$$\sum_{r=0}^{\infty} \frac{(-1)^r \sin\{(2r+1)x\}}{(2r+1)^{2n}} = \sum_{s=0}^{n-1} \frac{(-1)^s x^{2s+1}}{(2s+1)!} \beta(2n-1-2s)$$

$$\begin{aligned}\beta(2n) = {} & \frac{(-1)^n}{2(2n-1)!} \left(\frac{\pi}{2}\right)^{2n-1} \left(\log\frac{\pi}{4} - H_{2n-1}\right) \\ & + \frac{(-1)^n}{2} \sum_{r=1}^{\infty} \frac{\left(2^{2r}-2\right)|B_{2r}|}{2r(2r+2n-1)!} \left(\frac{\pi}{2}\right)^{2n-1+2r} \\ & - \sum_{s=1}^{n-1} \frac{(-1)^s}{(2s-1)!} \left(\frac{\pi}{2}\right)^{2s-1} \lambda(2n+1-2s)\end{aligned}$$

Furthermore, if the termwise higher order differentiation of the Fourier series of each family of sech and sec are carried out, the following expressions are obtained.

$$\beta(-n) = \frac{1}{2^{n+1}} \sum_{r=0}^{n} (-1)^r {}_nK_r \qquad n=1,2,3,\cdots$$

$$\beta(-2n) = \frac{1}{2^{2n+1}} \sum_{r=0}^{2n} (-1)^r {}_{2n}K_r \qquad n=1,2,3,\cdots$$

$$= \frac{E_{2n}}{2n} \qquad n=1,2,3,\cdots$$

Where, ${}_nK_r$ is a kind of Eulderian Number and is defined as follows.

$${}_nK_r = \sum_{k=0}^{r} (-1)^k \binom{n+1}{k} (2r+1-2k)^n \qquad n=1,2,3,\cdots$$

2 Formulas for Dirichlet Beta

Here, removing the lower betas from the the automorphism formulas in the previous chapter, we obtain the following non-automorphism formulas. Where, Bernoulli numbers and Euler numbers are as follows.

$$B_0=1,\ B_2=1/6,\ B_4=-1/30,\ B_6=1/42,\ B_8=-1/30,\cdots$$

$$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ \cdots$$

And, gamma function and incomplete gamma function were as follows.

$$\Gamma(p)=\int_0^\infty t^{p-1}e^{-t}dt \quad , \quad \Gamma(p,x)=\int_x^\infty t^{p-1}e^{-t}dt$$

2.1 Formulas for Beta at natural number

For $0<x\le\pi/2$,

$$\beta(n)=\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{(2r+1)^s x^s}{s!}\frac{(-1)^r e^{-(2r+1)x}}{(2r+1)^n}+\frac{x^n}{2}\sum_{r=0}^{\infty}\binom{-n}{2r}\frac{E_{2r}x^{2r}}{(n+2r)!}$$

Especially,

$$\beta(n)=\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{(2r+1)^s}{s!\,2^s}\frac{(-1)^r e^{-(r+1/2)}}{(2r+1)^n}+\frac{1}{2^{n+1}}\sum_{r=0}^{\infty}\binom{-n}{2r}\frac{E_{2r}}{(n+2r)!\,2^{2r}}$$

$$\beta(n)=\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{(2r+1)^s}{s!}\frac{(-1)^r e^{-(2r+1)}}{(2r+1)^n}+\frac{1}{2}\sum_{r=0}^{\infty}\binom{-n}{2r}\frac{E_{2r}}{(n+2r)!}$$

Example

$$\beta(4)=\sum_{r=0}^{\infty}\left\{1+\frac{2r+1}{1!\,2^1}+\frac{(2r+1)^2}{2!\,2^2}+\frac{(2r+1)^3}{3!\,2^3}\right\}\frac{(-1)^r e^{-r-\frac{1}{2}}}{(2r+1)^4}+\frac{1}{2^5}\sum_{r=0}^{\infty}\binom{-4}{2r}\frac{E_{2r}}{(4+2r)!\,2^{2r}}$$

$$\beta(4)=\sum_{r=0}^{\infty}\left\{1+\frac{2r+1}{1!}+\frac{(2r+1)^2}{2!}+\frac{(2r+1)^3}{3!}\right\}\frac{(-1)^r e^{-2r-1}}{(2r+1)^4}+\frac{1}{2}\sum_{r=0}^{\infty}\binom{-4}{2r}\frac{E_{2r}}{(4+2r)!}$$

2.2 Formulas for Beta at even number

For $0<x\le\pi/2$,

$$\beta(2n)=\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^{2n}}-\frac{(-1)^n}{2}\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{n-1}\binom{2n+2r}{2s}E_{2s}\right\}\frac{|E_{2r}|x^{2n+2r}}{(2n+2r)!}$$

$$\beta(2n)=-\frac{1}{x}\sum_{r=1}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^{2n+1}}+(-1)^n\frac{x^{2n}}{2}\sum_{r=1}^{\infty}\left\{\sum_{s=0}^{n-1}\frac{\left(2^{2s}-2\right)B_{2s}}{(2s)!\,(2n+1+2r-2s)!}\right\}\frac{|E_{2r}|x^{2r}}{2r}$$

Especially,

$$\beta(2n)=\frac{(-1)^{n-1}}{2}\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{n-1}\binom{2n+2r}{2s}E_{2s}\right\}\frac{|E_{2r}|}{(2n+2r)!}\left(\frac{\pi}{2}\right)^{2n+2r}$$

Example

$$\beta(4)=-\frac{1}{2}\sum_{r=0}^{\infty}\left\{\binom{4+2r}{0}E_0+\binom{4+2r}{2}E_2\right\}\frac{|E_{2r}|}{(4+2r)!}\left(\frac{\pi}{2}\right)^{4+2r}$$

$$\beta(6)=\frac{1}{2}\sum_{r=0}^{\infty}\left\{\binom{6+2r}{0}E_0+\binom{6+2r}{2}E_2+\binom{6+2r}{4}E_4\right\}\frac{|E_{2r}|}{(6+2r)!}\left(\frac{\pi}{2}\right)^{6+2r}$$

2.3 Formulas for Beta at odd number

For $0<x\le\pi/2$,

$$\beta(2n-1)=\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{|E_{2s}|\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\cos\{(2r+1)x\}}{(2r+1)^{2n-1}}$$

$$\beta(2n-1)=-\frac{1}{x}\sum_{r=0}^{\infty}\sum_{s=0}^{n-1}\frac{(-1)^s B_{2s}\left(2^{2s}-2\right)\{(2r+1)x\}^{2s}}{(2s)!}\frac{(-1)^r\sin\{(2r+1)x\}}{(2r+1)^{2n}}$$

Especially,

$$\beta(2n-1)=\frac{\pi}{4}\frac{|E_{2n-2}|}{(2n-2)!}\left(\frac{\pi}{2}\right)^{2n-2}$$

2.4 Formulas for Beta at complex number

When p is a complex number such that $p\neq 1,0,-1,-2,\cdots$,

For $x=u+vi$ s.t. $0<|x|\leqq 2\pi$, $u\geqq 0$,

$$\beta(p)=\sum_{r=0}^{\infty}\frac{\Gamma\{p,(2r+1)x\}}{\Gamma(p)}\frac{(-1)^r}{(2r+1)^p}+\frac{x^p}{2}\sum_{r=0}^{\infty}\binom{-p}{2r}\frac{E_{2r}x^{2r}}{\Gamma(p+1+2r)}$$

Especially,

$$\beta(p)=\sum_{r=0}^{\infty}\frac{\Gamma(p,2r+1)}{\Gamma(p)}\frac{(-1)^r}{(2r+1)^p}+\frac{1}{2}\sum_{r=0}^{\infty}\binom{-p}{2r}\frac{E_{2r}}{\Gamma(p+1+2r)}$$

3 Global definition of Dirichlet Beta and Generalized Euler Number

Diriclet beta function is defined on the whole complex plane with patches as follows.

$$\beta(p)=\begin{cases}\displaystyle\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{(2r-1)^p} & Re(p)\geqq 0\\ \displaystyle\left(\frac{2}{\pi}\right)^{1-p}\cos\frac{p\pi}{2}\Gamma(1-p)\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{(2r-1)^{1-p}} & Re(p)<0\end{cases}$$

This is inconvenient. so, we focus on the following sequence.

$${}_nB_r=\sum_{s=0}^{r}(-1)^{r-s}{}_rC_s\left(s-\frac{1}{2}\right)^n \qquad r=0,1,2,\cdots,n$$

Using this sequence, we can define Diriclet beta function on the whole complex plane as follows.

Definition 3.2.1

$$\beta(p)=\sum_{r=1}^{\infty}\frac{1}{2^{r+1}}\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}(2s-1)^{-p}$$

Furthermore, by using this sequence, Euler Number can be defined on the whole complex plane.

Definition 3.3.1

$$E_p=\sum_{r=1}^{\infty}\frac{1}{2^r}\sum_{s=1}^{r}(-1)^{s-1}\binom{r}{s}(2s-1)^p$$

4 Completed Dirichlet Beta

In **4.1**, symmetric functional equations are derived from functional equations.

Formula 4.1.1

$$\left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1}{2}+\frac{z}{2}\right)\beta(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{2-z}\Gamma\left(\frac{1}{2}+\frac{1-z}{2}\right)\beta(1-z)$$

$$\left(\frac{2}{\sqrt{\pi}}\right)^{\frac{3}{2}+z}\Gamma\left(\frac{3}{4}+\frac{z}{2}\right)\beta\left(\frac{1}{2}+z\right) = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{3}{2}-z}\Gamma\left(\frac{3}{4}-\frac{z}{2}\right)\beta\left(\frac{1}{2}-z\right)$$

In **4.2**, we define the completed Dirichlet beta functions $\omega(z)$, $\Omega(z)$ as follows, respectively.

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z)$$

$$\Omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{3}{2}+z}\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+z\right)\right\}\beta\left(\frac{1}{2}+z\right)$$

Then, Formula 4.1.1 is expressed as follows.

$$\omega(z) = \omega(1-z)$$

$$\Omega(z) = \Omega(-z)$$

From the latter, we can see that $\Omega(z)$ is an even function. Therefore, $\Omega(z)$ has the same properties as completed Riemann zeta function $\Xi(z)$. (See " 07 Completed Riemann Zeta ".) And, as in $\Xi(z)$, the following theorem holds.

Theorem 4.2.1

If Dirichlet beta function $\beta(z)$ has a non-trivial zero whose real part is not $1/2$, the one set consists of the following four.

$$1/2+\alpha_1 \pm i\delta_1 \quad , \quad 1/2-\alpha_1 \pm i\delta_1 \qquad (0 < \alpha_1 < 1/2)$$

05 Factorization of Completed Dirichlet Beta

In **5.1**, the following Hadamard product is derived.

Formula 5.1.1

Let completed beta function be as follows.

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z)$$

When non-trivial zeros of $\beta(z)$ are $z_k = x_k \pm iy_k$ $k=1,2,3,\cdots$ and γ is Euler-Mascheroni constant, $\omega(z)$ is expressed by the Hadamard product as follows.

$$\omega(z) = e^{\left(\frac{3\log\pi}{2}-\frac{\gamma}{2}-\log 2-4\log\Gamma\left(\frac{3}{4}\right)\right)z}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{\frac{z}{z_k}}$$

$$\omega(z) = e^{\left(\frac{3\log\pi}{2}-\frac{\gamma}{2}-\log 2-4\log\Gamma\left(\frac{3}{4}\right)\right)z}\prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)e^{\frac{2x_n z}{x_n^2+y_n^2}}$$

And, the following special values are obtained.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right)e^{\frac{2x_n}{x_n^2+y_n^2}} = e^{4\log\Gamma\left(\frac{3}{4}\right)+\frac{\gamma}{2}+\log 2-\frac{3\log\pi}{2}} = 1.08088915\cdots$$

$$\prod_{n=1}^{\infty}\left\{1-\frac{1}{(x_n+iy_n)^2}\right\}\left\{1-\frac{1}{(x_n-iy_n)^2}\right\} = \omega(-1) = 1.16624361\cdots$$

In **5.2**, we consider how the formulas in the previous section are expressed when non-trivial zeros whose real part is 1/2 and non-trivial zeros whose real part is not 1/2 are mixed. Then, we obtain the following theorems.

Theorem 5.2.2

Let γ be Euler-Mascheroni constant, non-trivial zeros of Dirichlet beta function are x_n+iy_n $n=1,2,3,\cdots$. Among them, zeros whose real part is 1/2 are $1/2\pm iy_r$ $r=1,2,3,\cdots$ and zeros whose real parts is not 1/2 are $1/2\pm\alpha_s\pm i\delta_s$ $(0<\alpha_s<1/2)$ $s=1,2,3,\cdots$. Then the following expressions hold.

$$\prod_{n=1}^{\infty}\left(1-\frac{2x_n-1}{x_n^2+y_n^2}\right) = 1$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2} = \sum_{r=1}^{\infty}\frac{1}{1/4+y_r^2}+\sum_{s=1}\left\{\frac{1+2\alpha_s}{(1/2+\alpha_s)^2+\delta_s^2}+\frac{1-2\alpha_s}{(1/2-\alpha_s)^2+\delta_s^2}\right\}$$

$$\sum_{n=1}^{\infty}\frac{2x_n}{x_n^2+y_n^2} = 4\log\Gamma\left(\frac{3}{4}\right)+\frac{\gamma}{2}+\log 2-\frac{3\log\pi}{2} = 0.07778398\cdots$$

Formula 5.2.3 (Special values)

When non-trivial zeros of Dirichlet beta function are $x_k\pm iy_k$ $k=1,2,3,\cdots$, the following expressions hold.

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{x_n+iy_n}\right)\left(1-\frac{1}{x_n-iy_n}\right) = 1$$

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{x_n+iy_n}\right)\left(1+\frac{1}{x_n-iy_n}\right) = \omega(-1) = 1.1662436\cdots$$

Theorem 5.2.4

Let non-trivial zeros of Dirichlet beta function are x_n+iy_n $n=1,2,3,\cdots$ and γ be Euler-Mascheroni constant. If the following expression holds, non-trivial zeros whose real parts is not $1/2$ do not exist.

$$\sum_{n=1}^{\infty}\frac{1}{1/4+y_n^2} = 4\log\Gamma\left(\frac{3}{4}\right)+\frac{\gamma}{2}+\log 2-\frac{3\log\pi}{2} = 0.07778398\cdots$$

Incidentally, when this was calculated using 10000 y_r , both sides coincided with the decimal point 3 digits.

In **5.3**, we show that $\omega(z)$ is factored completely.

Theorem 5.3.1 (Factorization of $\omega(z)$)

Let Dirichlet beta function be $\beta(z)$, the non-trivial zeros are $z_n=x_n\pm iy_n$ $n=1,2,3,\cdots$ and completed beta function be as follows.

$$\omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{1+z}\Gamma\left(\frac{1+z}{2}\right)\beta(z)$$

Then, $\omega(z)$ is factorized as follows.

$$\omega(z) = \prod_{n=1}^{\infty}\left(1-\frac{2x_n z}{x_n^2+y_n^2}+\frac{z^2}{x_n^2+y_n^2}\right)$$

In **5.4**, we first derive the factorization of $\Omega(z)$.

Theorem 5.4.1 (Factorization of $\Omega(z)$)

Let Dirichlet beta function be $\beta(z)$, the non-trivial zeros are $z_n = x_n \pm i y_n \quad n=1,2,3,\cdots$ and completed beta function be as follows.

$$\Omega(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{3}{2}+z} \Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+z\right)\right\}\beta\left(\frac{1}{2}+z\right)$$

Then, $\Omega(z)$ is factorized as follows.

$$\Omega(z) = \Omega(0)\prod_{n=1}^{\infty}\left\{1-\frac{2(x_n-1/2)z}{(x_n-1/2)^2+y_n^2}+\frac{z^2}{(x_n-1/2)^2+y_n^2}\right\}$$

Where, $\Omega(0) = \prod_{n=1}^{\infty}\frac{(x_n-1/2)^2+y_n^2}{x_n^2+y_n^2} = \left(\frac{2}{\sqrt{\pi}}\right)^{3/2}\Gamma\left(\frac{3}{4}\right)\beta\left(\frac{1}{2}\right) = 0.98071361\cdots$

And, using this theorem and Theorem 4.2.1 in the previous section, we obtaine the following theorem.

Theorem 5.4.4

When Dirichlet beta function is $\beta(z)$ and the non-trivial zeros are $z_n = x_n \pm i y_n \quad n=1,2,3,\cdots$,
If the following expression holds, non-trivial zeros whose real parts is not $1/2$ do not exist.

$$\prod_{r=1}^{\infty}\frac{y_r^2}{1/4+y_r^2} = \left(\frac{2}{\sqrt{\pi}}\right)^{3/2}\Gamma\left(\frac{3}{4}\right)\beta\left(\frac{1}{2}\right) = 0.98071361\cdots \qquad (4.4_0)$$

Incidentally, when this was calculated using 10000 y_r , both sides coincided with the decimal point 4 digits.

06 Zeros on the Critical Line of Dirichlet Beta

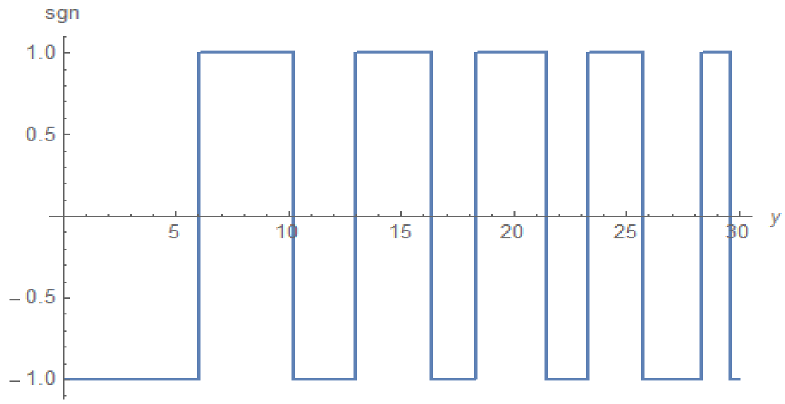
In **6.1**, substituting $z=0+iy$ for the completed Dirichlet beta $\Omega(z)$,

$$\Omega_h(z) = \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{3}{2}+iy} \Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\beta\left(\frac{1}{2}+iy\right)$$

We use this to calculate the zeros on the critical line. However, this function is too small in absolute value and can only find the zeros up to $y=917$.

So we normalize $\Omega_h(y)$ and define the following sign function.

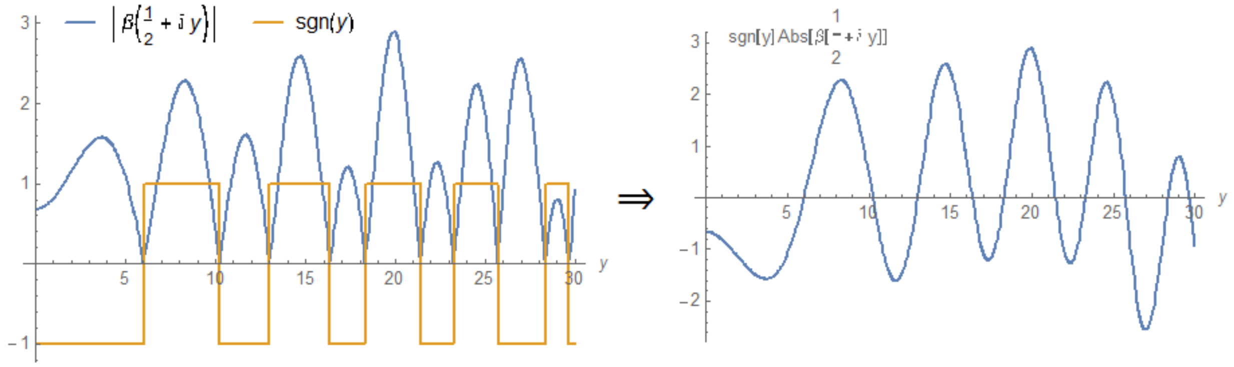
$$sgn(y) = -\frac{\Omega_h(y)}{|\Omega_h(y)|} = -\left(\frac{2}{\sqrt{\pi}}\right)^{iy}\frac{\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\beta\left(\frac{1}{2}+iy\right)}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\beta\left(\frac{1}{2}+iy\right)\right|}$$



Using this sign function $sgn(y)$, we can find the zeros at large y .

In **6.2**, multiplying this sign function $sgn(y)$ by the absolute value of the Dirichlet beta $\beta(1/2+iy)$, we obtain a smooth function $B(y)$.

$$B(y) = sgn(y)\left|\beta\left(\frac{1}{2}+iy\right)\right| = -\left(\frac{2}{\sqrt{\pi}}\right)^{iy}\frac{\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\right|}\beta\left(\frac{1}{2}+iy\right)$$



Using this $B(y)$ function, we can find the zeros on the critical line of $\beta(z)$ by the intersection of the curve and the y -axis

In **6.3**, first, a lemma is prepared.

Lemma

When $f(z)$ is a complex function defined on the domain D , the following expression holds.

$$e^{i\, Im \log f(z)} = \frac{f(z)}{|f(z)|}$$

Applying this lemma to the gamma function in the **6.2** ,

$$B(y) = -\left(\frac{2}{\sqrt{\pi}}\right)^{iy}\frac{\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}}{\left|\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}\right|}\beta\left(\frac{1}{2}+iy\right)$$

$$= -\left(\frac{2}{\sqrt{\pi}}\right)^{iy} e^{i\, Im \log \Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\}}\beta\left(\frac{1}{2}+iy\right)$$

$$= -e^{i\left[Im \log \Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\} - \frac{y}{2}\log\frac{\pi}{4}\right]}\beta\left(\frac{1}{2}+iy\right)$$

From this, we obtain

$$B(y) = -e^{i\theta(y)}\beta\left(\frac{1}{2}+iy\right) \qquad \text{where,} \quad \theta(y) = Im \log\Gamma\left\{\frac{1}{2}\left(\frac{3}{2}+iy\right)\right\} - \frac{y}{2}\log\frac{\pi}{4}$$

This is Riemann-Siegel style B function .

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⋮

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