欧拉积分

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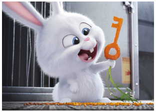
方向: ？？？？



《微积分》

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目录

Γ 函数定义

♦ 定义 2.1. Γ 函数

Γ 函数 (Gamma function) 的定义为

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}\mathrm{d}t \quad (\mathrm{Re}\, z > 0)$$

上式的右边称为第二类欧拉 (Euler) 积分

其它形式

$$\Gamma(z) = 2\int_0^{+\infty} e^{-t^2}t^{2z-1}\mathrm{d}t \quad (\mathrm{Re}\, z > 0)$$

Γ 函数定义

♦ 定义 2.2. Euler-Gauss 公式

$\forall z > 0$，有

$$\Gamma(z) = \lim_{m \to +\infty} \frac{m^z m!}{z(z+1)\cdots(z+m)} \quad (\mathrm{Re}\, z > 0)$$

Γ 函数定义

♦ 定义 2.3. Bohr-Mollerup 命题

如果定义在 $(0,+\infty)$ 上的函数 f 满足以下三个条件:

(1) $f(x)>0$, 且 $f(1)=1$,

(2) $f(x+1)=xf(x)$,

(3) $\ln f(x)$ 是 $(0,+\infty)$ 内的下凹函数

则 $f(x)\equiv\Gamma(x), x\in(0,+\infty)$

Γ 函数递推公式及相关公式

(1) 递推公式

$$\Gamma(z+1) = z\Gamma(z) \quad (z > 0)$$

(2) 余元公式: 对于 $0 < z < 1$ 有

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

(3) Legendre 加倍公式: 对于 $z > 0$ 有

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

(4) 对于 $z > 0$ 有

$$\Gamma(1-z) = -z\Gamma(-z)$$

(5) 对于 $0 < z < 1$

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z \sin \pi z}$$

Γ 函数递推公式及相关公式

(6)

$$\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right)=-\frac{\pi}{\cos \pi x}$$

(7)

$$\Gamma(3z)=\frac{3^{3z-\frac{1}{2}}}{2\pi}\Gamma(z)\Gamma\left(z+\frac{1}{3}\right)\Gamma\left(z+\frac{2}{3}\right)$$

(8)

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^n}(2n-1)!!=\frac{\sqrt{\pi}}{2^{2n}}\frac{(2n)!}{n!}$$

(9)

$$\Gamma\left(\frac{1}{2}-n\right)=(-1)^n\frac{2^n\sqrt{\pi}}{(2n-1)!!}$$

(10) *for* $\alpha_1+\alpha_2+\cdots+\alpha_n=\beta_1+\beta_2+\cdots+\beta_n$ then

$$\frac{\Gamma(\beta_1)\cdot\cdots\cdot\Gamma(\beta_n)}{\Gamma(\alpha_1)\cdot\cdots\cdot\Gamma(\alpha_n)}=\prod_{k\geqslant 0}\frac{(k+\alpha_1)\cdot\cdots\cdot(k+\alpha_n)}{(k+\beta_1)\cdot\cdots\cdot(k+\beta_n)}$$

Γ 函数递推公式及相关公式

(11)

$$\zeta(s)\Gamma(s) = \int_0^{+\infty} \frac{x^{s-1}}{e^x - 1} dx \quad s > 1$$

(12)

$$\Gamma\left(1 + \frac{1}{n}\right) \cos\left(\frac{\pi}{2n}\right) = \int_0^{+\infty} \cos(t^n) \mathrm{d}t \quad n = 2, 3, \cdots$$

(13)

$$\Gamma\left(1 + \frac{1}{n}\right) \sin\left(\frac{\pi}{2n}\right) = \int_0^{+\infty} \sin(t^n) \mathrm{d}t \quad n = 2, 3, \cdots$$

(14)

$$\Gamma(z) \cos\left(\frac{1}{2}\pi z\right) = \int_0^{+\infty} t^{z-1} \cos(t^n) \mathrm{d}t \quad 0 < \mathrm{Re}\, z < 1$$

(15)

$$\Gamma(z) \sin\left(\frac{1}{2}\pi z\right) = \int_0^{+\infty} t^{z-1} \sin(t^n) \mathrm{d}t \quad -1 < \mathrm{Re}\, z < 1$$

特殊值

(1) 一般地, 对于任何正整数 n 有 $\Gamma(n+1) = n!$

(2) $\Gamma(1) = \Gamma(2) = 1$

(3) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} = \left(-\frac{1}{2}\right)!$

(4) $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} = \left(\frac{1}{2}\right)!$

(5) $\Gamma\left(n+\frac{1}{4}\right) = \frac{\prod_{i=1}^{n}(4i-3)}{4^n}\Gamma\left(\frac{1}{4}\right) \quad (n = 1, 2, 3, \cdots)$

(6) $\Gamma\left(\frac{1}{4}\right) \approx 3.6256099082\cdots$

(7) $\Gamma\left(n+\frac{1}{3}\right) = \frac{\prod_{i=1}^{n}(3i-2)}{3^n}\Gamma\left(\frac{1}{3}\right) \quad (n = 1, 2, 3, \cdots)$

(8) $\Gamma\left(\frac{1}{3}\right) \approx 2.6789385347\cdots$

特殊值

(9) $\Gamma\left(n+\frac{1}{2}\right)=\dfrac{\prod\limits_{i=1}^{n}(2i-1)}{2^n}\Gamma\left(\dfrac{1}{2}\right)\quad(n=1,2,3,\cdots)$

(10 $\Gamma\left(n+\frac{2}{3}\right)=\dfrac{\prod\limits_{i=1}^{n}(3i-1)}{3^n}\Gamma\left(\dfrac{2}{3}\right)\quad(n=1,2,3,\cdots)$

(11) $\Gamma\left(\frac{2}{3}\right)\approx 1.3541179394\cdots$

(12) $\Gamma\left(n+\frac{3}{4}\right)=\dfrac{\prod\limits_{i=1}^{n}(4i-1)}{4^n}\Gamma\left(\dfrac{3}{4}\right)\quad(n=1,2,3,\cdots)$

(13) $\Gamma\left(\frac{3}{4}\right)\approx 1.2254167024\cdots$

(14) $\prod\limits_{k=1}^{n-1}\Gamma\left(\dfrac{k}{n}\right)\Gamma\left(1-\dfrac{k}{n}\right)=\dfrac{(2\pi)^{n-1}}{n}$

(15) $\Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi}$

(16) $\Gamma'(1)=-\gamma$

Γ 函数的几个例题

例 1 计算积分 $\int_0^{+\infty} e^{-x^2}\mathrm{d}x$

解

$$\int_0^{+\infty} e^{-x^2}dx \xlongequal[dx=\frac{1}{2\sqrt{t}}dt]{x^2=t} \frac{1}{2}\int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}dt = \frac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}$$

例 2 计算积分 $\int_0^{+\infty} x^{2n-1}e^{-x^2}\mathrm{d}x$

解

$$\int_0^{+\infty} x^{2n-1}e^{-x^2}dx \xlongequal[dx=\frac{1}{2\sqrt{t}}dt]{x^2=t} \frac{1}{2}\int_0^{+\infty} x^{n-1}e^{-t}dt$$

$$= \frac{1}{2}\Gamma(n) = \frac{1}{2}(n-1)!$$

Γ 函数的几个例题

例 1 计算积分 $\int_0^{+\infty} e^{-x^2}\mathrm{d}x$

解

$$\int_0^{+\infty} e^{-x^2}dx \xlongequal[dx=\frac{1}{2\sqrt{t}}dt]{x^2=t} \frac{1}{2}\int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}dt = \frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

例 2 计算积分 $\int_0^{+\infty} x^{2n-1}e^{-x^2}\mathrm{d}x$

解

$$\int_0^{+\infty} x^{2n-1}e^{-x^2}dx \xlongequal[dx=\frac{1}{2\sqrt{t}}dt]{x^2=t} \frac{1}{2}\int_0^{+\infty} x^{n-1}e^{-t}dt$$

$$= \frac{1}{2}\Gamma(n) = \frac{1}{2}(n-1)!$$

Γ 函数的几个例题

例 1 计算积分 $\int_0^{+\infty} e^{-x^2}\mathrm{d}x$

解

$$\int_0^{+\infty} e^{-x^2}dx \xlongequal[dx=\frac{1}{2\sqrt{t}}dt]{x^2=t} \frac{1}{2}\int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}dt = \frac{1}{2}\Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2}$$

例 2 计算积分 $\int_0^{+\infty} x^{2n-1}e^{-x^2}\mathrm{d}x$

解

$$\int_0^{+\infty} x^{2n-1}e^{-x^2}dx \xlongequal[dx=\frac{1}{2\sqrt{t}}dt]{x^2=t} \frac{1}{2}\int_0^{+\infty} x^{n-1}e^{-t}dt$$

$$= \frac{1}{2}\Gamma(n) = \frac{1}{2}(n-1)!$$

Γ 函数的几个例题

例 3 计算积分 $\int_0^{+\infty} \frac{1-e^{-x^2}}{x^2} \mathrm{d}x$

解

$$
\begin{aligned}
\int_0^{+\infty} \frac{1-e^{-x^2}}{x^2} \mathrm{d}x &= \int_0^{+\infty} (e^{-x^2}-1) \mathrm{d}\left(\frac{1}{x}\right) \\
&= \frac{e^{-x^2}-1}{x}\Bigg|_0^{+\infty} - \int_0^{+\infty} \frac{1}{x} \cdot e^{-x^2} \cdot (-2x) \mathrm{d}x \\
&= 0 + 2\int_0^{+\infty} e^{-x^2} \mathrm{d}x \\
&= \int_0^{+\infty} (x^2)^{-\frac{1}{2}} \cdot e^{-x^2} \mathrm{d}(x^2) \\
&= \Gamma(\tfrac{1}{2}) = \sqrt{\pi}
\end{aligned}
$$

Γ 函数的几个例题

例 3 计算积分 $\displaystyle\int_0^{+\infty}\frac{1-e^{-x^2}}{x^2}\mathrm{d}x$

解

$$\begin{aligned}\int_0^{+\infty}\frac{1-e^{-x^2}}{x^2}\mathrm{d}x &= \int_0^{+\infty}(e^{-x^2}-1)\mathrm{d}\left(\frac{1}{x}\right)\\ &= \frac{e^{-x^2}-1}{x}\bigg|_0^{+\infty}-\int_0^{+\infty}\frac{1}{x}\cdot e^{-x^2}\cdot(-2x)\mathrm{d}x\\ &= 0+2\int_0^{+\infty}e^{-x^2}\mathrm{d}x\\ &= \int_0^{+\infty}(x^2)^{-\frac{1}{2}}\cdot e^{-x^2}\mathrm{d}(x^2)\\ &= \Gamma(\tfrac{1}{2})=\sqrt{\pi}\end{aligned}$$

Γ 函数的几个例题

例 4 计算积分 $\int_0^1 \frac{1}{\sqrt{1+x^4}}\mathrm{d}x$

解 我们有

$$\begin{aligned} J &= \int_0^{+\infty} \frac{1}{\sqrt{1+x^4}}\mathrm{d}x \\ &\xlongequal{x^4=t} \frac{1}{4}\int_0^{+\infty} \frac{t^{-\frac{3}{4}}}{(1+t)^{\frac{1}{2}}}\mathrm{d}t \\ &= \frac{1}{4}B(\frac{1}{4},\frac{1}{4}) = \frac{1}{4}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{2})} \\ &= \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}} \end{aligned}$$

接着我们对积分 $\int_1^{+\infty} \frac{1}{\sqrt{1+x^4}}\mathrm{d}x$ 做变量替换

Γ 函数的几个例题

例 4 计算积分 $\int_0^1 \frac{1}{\sqrt{1+x^4}}\mathrm{d}x$

解 我们有

$$\begin{aligned}
J &= \int_0^{+\infty} \frac{1}{\sqrt{1+x^4}}\mathrm{d}x \\
&\xlongequal{x^4=t} \frac{1}{4}\int_0^{+\infty} \frac{t^{-\frac{3}{4}}}{(1+t)^{\frac{1}{2}}}\mathrm{d}t \\
&= \frac{1}{4}B(\frac{1}{4},\frac{1}{4}) = \frac{1}{4}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{2})} \\
&= \frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}
\end{aligned}$$

接着我们对积分 $\int_1^{+\infty} \frac{1}{\sqrt{1+x^4}}\mathrm{d}x$ 做变量替换

Γ 函数的几个例题

令 $t = \dfrac{1}{x}$，可得

$$\int_1^{+\infty} \frac{1}{\sqrt{1+x^4}} \mathrm{d}x = \int_0^1 \frac{1}{\sqrt{1+t^4}} \mathrm{d}t$$

由此知

$$\begin{aligned} J &= \int_0^1 \frac{1}{\sqrt{1+x^4}} dx + \int_1^{+\infty} \frac{1}{\sqrt{1+x^4}} dx \\ &= 2\int_0^1 \frac{1}{\sqrt{1+t^4}} dt \\ &= 2I \end{aligned}$$

所以

$$I = \int_0^1 \frac{1}{\sqrt{1+x^4}} dx = \frac{J}{2} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}}$$

Γ 函数的几个例题

例 5 计算积分 $\int_0^{+\infty} e^{-(ax^2+bx)}\mathrm{d}x$

解

$$
\begin{aligned}
I &= \int_0^{+\infty} e^{-(ax^2+bx)}\mathrm{d}x \\
&= \int_0^{+\infty} e^{-a\left[\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right)-\left(\frac{b}{2a}\right)^2\right]}\mathrm{d}x \\
&= e^{\frac{b^2}{4a}}\frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-\left(\sqrt{a}\left(x+\frac{b}{2a}\right)\right)^2}\mathrm{d}\left(\sqrt{a}\left(x+\frac{b}{2a}\right)\right) \\
&= e^{\frac{b^2}{4a}}\frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-u^2}\mathrm{d}u = e^{\frac{b^2}{4a}}\frac{1}{2\sqrt{a}}\int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}dt \\
&= e^{\frac{b^2}{4a}}\frac{1}{\sqrt{a}}\cdot\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = e^{\frac{b^2}{4a}}\frac{1}{\sqrt{a}}\cdot\frac{\sqrt{\pi}}{2} \\
&= \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}
\end{aligned}
$$

Γ 函数的几个例题

例 5 计算积分 $\int_0^{+\infty} e^{-(ax^2+bx)}\mathrm{d}x$

解

$$
\begin{aligned}
I &= \int_0^{+\infty} e^{-(ax^2+bx)}\mathrm{d}x \\
&= \int_0^{+\infty} e^{-a\left[\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right)-\left(\frac{b}{2a}\right)^2\right]}\mathrm{d}x \\
&= e^{\frac{b^2}{4a}}\frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-\left(\sqrt{a}\left(x+\frac{b}{2a}\right)\right)^2}\mathrm{d}\left(\sqrt{a}\left(x+\frac{b}{2a}\right)\right) \\
&= e^{\frac{b^2}{4a}}\frac{1}{\sqrt{a}}\int_0^{+\infty} e^{-u^2}\mathrm{d}u = e^{\frac{b^2}{4a}}\frac{1}{2\sqrt{a}}\int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}dt \\
&= e^{\frac{b^2}{4a}}\frac{1}{\sqrt{a}}\cdot\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = e^{\frac{b^2}{4a}}\frac{1}{\sqrt{a}}\cdot\frac{\sqrt{\pi}}{2} \\
&= \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}
\end{aligned}
$$

Γ 函数的几个例题

例 6 计算积分 $\int_0^{+\infty} e^{-\left(x^2+\frac{a^2}{x^2}\right)}\mathrm{d}x,\ a>0$

解

$$
\begin{aligned}
I &= \int_0^{+\infty} e^{-\left(x^2+\frac{a^2}{x^2}\right)}\mathrm{d}x \\
&= \underbrace{\int_0^{\sqrt{a}} e^{-\left(x^2+\frac{a^2}{x^2}\right)}\mathrm{d}x}_{t=\frac{a}{x}} + \int_{\sqrt{a}}^{+\infty} e^{-\left(x^2+\frac{a^2}{x^2}\right)}\mathrm{d}x \\
&= \int_{\sqrt{a}}^{+\infty} \frac{a}{t^2} e^{-\left(t^2+\frac{a^2}{t^2}\right)}\mathrm{d}t + \int_{\sqrt{a}}^{+\infty} e^{-\left(x^2+\frac{a^2}{x^2}\right)}\mathrm{d}x \\
&= \int_{\sqrt{a}}^{+\infty} \left(1+\frac{a}{x^2}\right) e^{-\left(x-\frac{a}{x}\right)^2-2a}\mathrm{d}x \\
&\xlongequal{u=x-\frac{a}{x}} e^{-2a}\int_0^{+\infty} e^{-u^2}\mathrm{d}u = \frac{1}{2}e^{-2a}\int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}\mathrm{d}t \\
&= \frac{1}{2}e^{-2a}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}e^{-2a}
\end{aligned}
$$

Γ 函数的几个例题

例 6 计算积分 $\int_0^{+\infty} e^{-\left(x^2+\frac{a^2}{x^2}\right)}\mathrm{d}x,\ a>0$

解

$$
\begin{aligned}
I &= \int_0^{+\infty} e^{-\left(x^2+\frac{a^2}{x^2}\right)}\mathrm{d}x \\
&= \underbrace{\int_0^{\sqrt{a}} e^{-\left(x^2+\frac{a^2}{x^2}\right)}\mathrm{d}x}_{t=\frac{a}{x}} + \int_{\sqrt{a}}^{+\infty} e^{-\left(x^2+\frac{a^2}{x^2}\right)}\mathrm{d}x \\
&= \int_{\sqrt{a}}^{+\infty} \frac{a}{t^2} e^{-\left(t^2+\frac{a^2}{t^2}\right)}\mathrm{d}t + \int_{\sqrt{a}}^{+\infty} e^{-\left(x^2+\frac{a^2}{x^2}\right)}\mathrm{d}x \\
&= \int_{\sqrt{a}}^{+\infty} \left(1+\frac{a}{x^2}\right) e^{-\left(x-\frac{a}{x}\right)^2-2a}\mathrm{d}x \\
&\xlongequal{u=x-\frac{a}{x}} e^{-2a}\int_0^{+\infty} e^{-u^2}\mathrm{d}u = \frac{1}{2}e^{-2a}\int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}\mathrm{d}t \\
&= \frac{1}{2}e^{-2a}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2}e^{-2a}
\end{aligned}
$$

Γ 函数的几个例题

例 7 **计算积分** $\int_0^{+\infty} e^{-\left(ax^2+\frac{b}{x^2}\right)}\mathrm{d}x$

解

$$
\begin{aligned}
I &= \int_0^{+\infty} e^{-\left(ax^2+\frac{b}{x^2}\right)}\mathrm{d}x = \int_0^{+\infty} e^{-a\left(x^2+\frac{\frac{b}{a}}{x^2}\right)}\mathrm{d}x \\
&= \int_0^{+\infty} e^{-a\left[\left(x-\frac{\sqrt{\frac{b}{a}}}{x}\right)^2+2\sqrt{\frac{b}{a}}\right]}\mathrm{d}x \\
&= \int_0^{+\infty} e^{-a\left(t^2+2\sqrt{\frac{b}{a}}\right)}\mathrm{d}t = e^{-2\sqrt{ab}}\int_0^{+\infty} e^{-ax^2}\mathrm{d}t \\
&\xlongequal{t=ax^2} e^{-2\sqrt{ab}}\cdot\frac{1}{2\sqrt{a}}\int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}\mathrm{d}t \\
&= \frac{1}{2\sqrt{a}}e^{-2\sqrt{ab}}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{a}}e^{-2\sqrt{ab}}
\end{aligned}
$$

公式: $\int_0^{+\infty} f\left(x-\frac{a}{x}\right)\mathrm{d}x = \int_0^{+\infty} f(x)\mathrm{d}x,\quad a>0$

Γ 函数的几个例题

例 7 计算积分 $\int_0^{+\infty} e^{-\left(ax^2+\frac{b}{x^2}\right)}\mathrm{d}x$

解

$$\begin{aligned}
I &= \int_0^{+\infty} e^{-\left(ax^2+\frac{b}{x^2}\right)}\mathrm{d}x = \int_0^{+\infty} e^{-a\left(x^2+\frac{\frac{b}{a}}{x^2}\right)}\mathrm{d}x \\
&= \int_0^{+\infty} e^{-a\left[\left(x-\frac{\sqrt{\frac{b}{a}}}{x}\right)^2+2\sqrt{\frac{b}{a}}\right]}\mathrm{d}x \\
&= \int_0^{+\infty} e^{-a\left(t^2+2\sqrt{\frac{b}{a}}\right)}\mathrm{d}t = e^{-2\sqrt{ab}}\int_0^{+\infty} e^{-ax^2}\mathrm{d}t \\
&\xlongequal{t=ax^2} e^{-2\sqrt{ab}}\cdot\frac{1}{2\sqrt{a}}\int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}\mathrm{d}t \\
&= \frac{1}{2\sqrt{a}}e^{-2\sqrt{ab}}\Gamma\left(\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{a}}e^{-2\sqrt{ab}}
\end{aligned}$$

公式: $\int_0^{+\infty} f\left(x-\frac{a}{x}\right)\mathrm{d}x = \int_0^{+\infty} f(x)\mathrm{d}x,\quad a>0$

Γ 函数的几个例题

例 8 计算积分 $\int_{-\infty}^{+\infty} e^{-\frac{x^2-Dx}{2}}\,\mathrm{d}x$

解

$$
\begin{aligned}
I &= \int_0^{+\infty} e^{-\frac{x^2-Dx}{2}}\,\mathrm{d}x + \int_{-\infty}^{0} e^{-\frac{x^2-Dx}{2}}\,\mathrm{d}x \\
&= \int_0^{+\infty} e^{-\frac{\left(x-\frac{D}{2}\right)^2-\frac{D^2}{4}}{2}}\,\mathrm{d}x + \int_0^{+\infty} e^{-\frac{t^2+Dt}{2}}\,\mathrm{d}t \\
&= \sqrt{2}e^{\frac{D^2}{8}} \int_0^{+\infty} e^{-\left(\frac{x}{\sqrt{2}}-\frac{D}{2\sqrt{2}}\right)^2}\,\mathrm{d}\left(\frac{x}{\sqrt{2}}-\frac{D}{2\sqrt{2}}\right) \\
&\quad + \sqrt{2}e^{\frac{D^2}{8}} \int_0^{+\infty} e^{-\left(\frac{t}{\sqrt{2}}+\frac{D}{2\sqrt{2}}\right)^2}\,\mathrm{d}\left(\frac{t}{\sqrt{2}}+\frac{D}{2\sqrt{2}}\right) \\
&= \sqrt{2}e^{\frac{D^2}{8}} \int_0^{+\infty} e^{-u^2}\,\mathrm{d}u + \sqrt{2}e^{\frac{-v^2}{8}} \int_0^{+\infty} e^{v^2}\,\mathrm{d}v \\
&= \sqrt{2\pi}e^{\frac{D^2}{8}}
\end{aligned}
$$

Γ 函数的几个例题

例 8 计算积分 $\int_{-\infty}^{+\infty} e^{-\frac{x^2-Dx}{2}}\,\mathrm{d}x$

解

$$\begin{aligned}
I &= \int_0^{+\infty} e^{-\frac{x^2-Dx}{2}}\,\mathrm{d}x + \int_{-\infty}^{0} e^{-\frac{x^2-Dx}{2}}\,\mathrm{d}x \\
&= \int_0^{+\infty} e^{-\frac{\left(x-\frac{D}{2}\right)^2-\frac{D^2}{4}}{2}}\,\mathrm{d}x + \int_0^{+\infty} e^{-\frac{t^2+Dt}{2}}\,\mathrm{d}t \\
&= \sqrt{2}e^{\frac{D^2}{8}} \int_0^{+\infty} e^{-\left(\frac{x}{\sqrt{2}}-\frac{D}{2\sqrt{2}}\right)^2}\,\mathrm{d}\left(\frac{x}{\sqrt{2}}-\frac{D}{2\sqrt{2}}\right) \\
&\quad + \sqrt{2}e^{\frac{D^2}{8}} \int_0^{+\infty} e^{-\left(\frac{t}{\sqrt{2}}+\frac{D}{2\sqrt{2}}\right)^2}\,\mathrm{d}\left(\frac{t}{\sqrt{2}}+\frac{D}{2\sqrt{2}}\right) \\
&= \sqrt{2}e^{\frac{D^2}{8}} \int_0^{+\infty} e^{-u^2}\,\mathrm{d}u + \sqrt{2}e^{\frac{-v^2}{8}} \int_0^{+\infty} e^{v^2}\,\mathrm{d}v \\
&= \sqrt{2\pi}e^{\frac{D^2}{8}}
\end{aligned}$$

Γ 函数的几个例题

例 9 计算

$$\int_0^{+\infty} \frac{x^3}{e^x - 1}\,\mathrm{d}x$$

解

$$\begin{aligned}
\int_0^{+\infty} \frac{x^3}{e^x - 1}\,\mathrm{d}x &= \int_0^{+\infty} x^3 \left(\sum_{n=1}^{\infty} e^{-nx}\right) \mathrm{d}x \\
&= \sum_{n=1}^{\infty} \int_0^{+\infty} x^3 e^{-nx}\,\mathrm{d}x \\
&= \sum_{n=1}^{\infty} \frac{1}{n^4} \int_0^{+\infty} t^3 e^{-t}\,\mathrm{d}t\,,\ t = nx \\
&= \sum_{n=1}^{\infty} \frac{1}{n^4}\Gamma(4) = 16 \sum_{n=1}^{\infty} \frac{1}{n^4} \\
&= 16 \times \frac{\pi^4}{90} = \frac{\pi^4}{15}
\end{aligned}$$

Γ 函数的几个例题

例 9 计算

$$\int_0^{+\infty} \frac{x^3}{e^x - 1}\,\mathrm{d}x$$

解

$$\begin{aligned}
\int_0^{+\infty} \frac{x^3}{e^x - 1}\,\mathrm{d}x &= \int_0^{+\infty} x^3 \left(\sum_{n=1}^{\infty} e^{-nx}\right)\,\mathrm{d}x \\
&= \sum_{n=1}^{\infty} \int_0^{+\infty} x^3 e^{-nx}\,\mathrm{d}x \\
&= \sum_{n=1}^{\infty} \frac{1}{n^4} \int_0^{+\infty} t^3 e^{-t}\,\mathrm{d}t\ ,\ t = nx \\
&= \sum_{n=1}^{\infty} \frac{1}{n^4} \Gamma(4) = 16 \sum_{n=1}^{\infty} \frac{1}{n^4} \\
&= 16 \times \frac{\pi^4}{90} = \frac{\pi^4}{15}
\end{aligned}$$

Γ 函数的几个例题

例 10 计算

$$\int_0^1 \sin(\pi x)\log\Gamma(x)\mathrm{d}x$$

解

$$I=\int_0^1 \sin(\pi x)\log\Gamma(x)dx \xlongequal{t=1-x} -\int_1^0 \sin(t\pi)\log\Gamma(1-t)dt$$

$$=\int_0^1 \sin(t\pi)\log\Gamma(1-t)dt$$

$$I=\frac{1}{2}\left(\int_0^1 \sin(\pi x)\log\Gamma(x)dx+\int_0^1 \sin(x\pi)\log\Gamma(1-x)dx\right)$$

$$=\frac{1}{2}\int_0^1 \sin(\pi x)\log\left(\Gamma(x)+\Gamma(1-x)\right)dx$$

$$=\frac{1}{2}\int_0^1 \sin(\pi x)\log\left(\frac{\pi}{\sin\pi x}\right)dx$$

$$=\frac{1}{\pi}\left(1+\ln\frac{\pi}{2}\right)$$

Γ 函数的几个例题

例 10 计算

$$\int_0^1 \sin(\pi x)\log\Gamma(x)\mathrm{d}x$$

解

$$I=\int_0^1 \sin(\pi x)\log\Gamma(x)dx \xlongequal{t=1-x} -\int_1^0 \sin(t\pi)\log\Gamma(1-t)dt$$

$$=\int_0^1 \sin(t\pi)\log\Gamma(1-t)dt$$

$$I=\frac{1}{2}\left(\int_0^1 \sin(\pi x)\log\Gamma(x)dx+\int_0^1 \sin(x\pi)\log\Gamma(1-x)dx\right)$$

$$=\frac{1}{2}\int_0^1 \sin(\pi x)\log\left(\Gamma(x)+\Gamma(1-x)\right)dx$$

$$=\frac{1}{2}\int_0^1 \sin(\pi x)\log\left(\frac{\pi}{\sin \pi x}\right)dx$$

$$=\frac{1}{\pi}\left(1+\ln\frac{\pi}{2}\right)$$

Γ 函数的几个例题

例 11 计算

$$\int_{\alpha}^{\alpha+1} \ln\Gamma(x)\,\mathrm{d}x$$

解 易得

$$\int_0^1 \ln\Gamma(x)\,\mathrm{d}x = \frac{1}{2}\ln(2\pi)$$

因此

$$\begin{aligned}\int_{\alpha}^{\alpha+1} \ln\Gamma(x)\,\mathrm{d}x &= \int_{\alpha}^{0} \ln\Gamma(x)\,\mathrm{d}x + \int_0^1 \ln\Gamma(x)\,\mathrm{d}x + \int_1^{\alpha+1} \ln\Gamma(x)\,\mathrm{d}x \\ &= -\frac{1}{2}\ln(2\pi) + \int_1^{\alpha+1} \ln\Gamma(x)\,\mathrm{d}x - \int_0^{\alpha} \ln\Gamma(x)\,\mathrm{d}x\end{aligned}$$

Γ 函数的几个例题

例 11 计算

$$\int_{\alpha}^{\alpha+1} \ln\Gamma(x)\,\mathrm{d}x$$

解 易得

$$\int_0^1 \ln\Gamma(x)\,\mathrm{d}x = \frac{1}{2}\ln(2\pi)$$

因此

$$\begin{aligned}\int_{\alpha}^{\alpha+1} \ln\Gamma(x)\,\mathrm{d}x &= \int_{\alpha}^{0} \ln\Gamma(x)\,\mathrm{d}x + \int_0^1 \ln\Gamma(x)\,\mathrm{d}x + \int_1^{\alpha+1} \ln\Gamma(x)\,\mathrm{d}x \\ &= -\frac{1}{2}\ln(2\pi) + \int_1^{\alpha+1} \ln\Gamma(x)\,\mathrm{d}x - \int_0^{\alpha} \ln\Gamma(x)\,\mathrm{d}x\end{aligned}$$

Γ 函数的几个例题

对上式右端第一个积分作变换:$x=1+t$, 得

$$\begin{aligned}\int_1^{\alpha+1}\ln\Gamma(x)\,\mathrm{d}x &= \int_0^{\alpha}\ln\Gamma(1+t)\,\mathrm{d}t=\int_0^{\alpha}\ln(t\Gamma(t))\,\mathrm{d}t\\ &=\int_0^{\alpha}\ln t\mathrm{d}t+\int_0^{\alpha}\ln\Gamma(t)\mathrm{d}t\\ &=\alpha\ln\alpha-\alpha+\int_0^{\alpha}\ln\Gamma(t)\mathrm{d}t\end{aligned}$$

于是有

$$\int_{\alpha}^{\alpha+1}\ln\Gamma(x)\,\mathrm{d}x=\frac{1}{2}\ln(2\pi)+\alpha\ln\alpha-\alpha$$

ψ 函数

♦ 定义 3.1. ψ 函数

ψ 函数 (Psi Function) 的定义为

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

ψ 函数

有关公式

- $\psi(x) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+x}\right)$ γ 为欧拉常数
- $\psi(x+1) = \psi(x) + \frac{1}{x}$
- $\psi(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k},\ n \in \mathbb{N}$
- $\psi(1-x) - \psi(x) = \pi\cot\pi x$
- $\psi(x) + \psi\left(x + \frac{1}{2}\right) - 2\ln 2 = 2\psi(2x)$
- $\psi\left(\frac{p}{q}\right) = -C + \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{q}{p+kq}\right)$

ψ 函数

ψ 函数的特殊值

- $\psi(1) = -\gamma$
- $\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2$
- $\psi\left(\frac{1}{4}\right) = -\gamma - \frac{\pi}{2} - 3\ln 2$
- $\psi\left(\frac{3}{4}\right) = -\gamma + \frac{\pi}{2} - 3\ln 2$
- $\psi\left(\frac{1}{6}\right) = -\gamma - \frac{\sqrt{3}\pi}{2} - \frac{3\ln 3}{2} - 2\ln 2$
- $\psi\left(\frac{5}{6}\right) = -\gamma + \frac{\sqrt{3}\pi}{2} - \frac{3\ln 3}{2} - 2\ln 2$
- $\psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2}$
- $\psi'\left(\frac{1}{4}\right) = 8\gamma + \pi^2$

ψ 函数的几个例题

例 1 计算

$$\lim_{n\to 0}\sqrt[n]{n!}$$

解

$$\begin{aligned}\lim_{n\to 0}\sqrt[n]{n!} &= \lim_{n\to 0}\exp\left\{\frac{\ln(n!)}{n}\right\}\\ &= \exp\left\{\lim_{n\to 0}\frac{\ln\Gamma(n+1)}{n}\right\}\\ &= \exp\left\{\lim_{x\to 0^+}\frac{\ln\Gamma(x+1)}{x}\right\}\\ &= \exp\left\{\lim_{x\to 0^+}\frac{\Gamma'(x+1)}{\Gamma(x+1)}\right\}\\ &= e^{\psi(1)} = e^{-\gamma}\end{aligned}$$

ψ 函数的几个例题

例 1 计算

$$\lim_{n\to 0}\sqrt[n]{n!}$$

解

$$\begin{aligned}\lim_{n\to 0}\sqrt[n]{n!} &= \lim_{n\to 0}\exp\left\{\frac{\ln(n!)}{n}\right\} \\ &= \exp\left\{\lim_{n\to 0}\frac{\ln\Gamma(n+1)}{n}\right\} \\ &= \exp\left\{\lim_{x\to 0+}\frac{\ln\Gamma(x+1)}{x}\right\} \\ &= \exp\left\{\lim_{x\to 0+}\frac{\Gamma'(x+1)}{\Gamma(x+1)}\right\} \\ &= e^{\psi(1)} = e^{-\gamma}\end{aligned}$$

ψ 函数的几个例题

例 2 求极限 $\lim\limits_{x\to 0}\dfrac{\Gamma(x+1)-\sin x\Gamma(\sin x)}{x^4\Gamma(\sin x)}$

解

$$
\begin{aligned}
I &= \lim_{x\to 0}\frac{\Gamma(x+1)-\sin x\Gamma(\sin x)}{x^4\Gamma(\sin x)} \\
&= \lim_{x\to 0}\frac{\Gamma(x+1)-\Gamma(\sin x+1)}{x^3\Gamma(\sin x+1)} \\
&= \lim_{x\to 0}\frac{\Gamma'(\xi+1)(x-\sin x)}{x^3} \quad \xi \text{ 介于 } \sin x \text{ 和 } x \text{ 之间} \\
&= \frac{\Gamma'(1)}{6} = -\frac{\gamma}{6}
\end{aligned}
$$

ψ 函数的几个例题

例 2 求极限 $\lim\limits_{x\to 0}\dfrac{\Gamma(x+1)-\sin x\Gamma(\sin x)}{x^4\Gamma(\sin x)}$

解

$$\begin{aligned} I &= \lim_{x\to 0}\frac{\Gamma(x+1)-\sin x\Gamma(\sin x)}{x^4\Gamma(\sin x)} \\ &= \lim_{x\to 0}\frac{\Gamma(x+1)-\Gamma(\sin x+1)}{x^3\Gamma(\sin x+1)} \\ &= \lim_{x\to 0}\frac{\Gamma'(\xi+1)(x-\sin x)}{x^3} \qquad \xi \text{ 介于 } \sin x \text{ 和 } x \text{ 之间} \\ &= \frac{\Gamma'(1)}{6} = -\frac{\gamma}{6} \end{aligned}$$

ψ 函数的几个例题

例 3 证明

$$\int_0^1 \frac{1-x^{z-1}}{1-x}\,\mathrm{d}x \quad \mathrm{Re}(z)>0$$

解 注意到

$$\frac{1-x^{z-1}}{1-x}=\sum_{k=1}^{\infty}(x^{k-1}-x^{k+z+2})$$

故

$$\begin{aligned}\int_0^1 \frac{1-x^{z-1}}{1-x}\,\mathrm{d}x &= \sum_{k=1}^{\infty}\int_0^1 (x^{k-1}-x^{k+z+2})\,\mathrm{d}x \\ &= -\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+z}\right) \\ &= \gamma+\psi(z+1)-\frac{1}{z} \\ &= \gamma+\psi(x)\end{aligned}$$

ψ 函数的几个例题

例 3 证明

$$\int_0^1 \frac{1-x^{z-1}}{1-x}\,\mathrm{d}x \quad \mathrm{Re}(z) > 0$$

解 注意到

$$\frac{1-x^{z-1}}{1-x} = \sum_{k=1}^{\infty}(x^{k-1} - x^{k+z+2})$$

故

$$\begin{aligned}
\int_0^1 \frac{1-x^{z-1}}{1-x}\,\mathrm{d}x &= \sum_{k=1}^{\infty}\int_0^1 (x^{k-1} - x^{k+z+2})\,\mathrm{d}x \\
&= -\frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+z}\right) \\
&= \gamma + \psi(z+1) - \frac{1}{z} \\
&= \gamma + \psi(x)
\end{aligned}$$

β 函数

♦ 定义 4.1. β 函数

β 函数的定义为

$$\beta(x) = \frac{1}{2}\left[\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right]$$

Integral representations:

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}\mathrm{d}t \quad \mathrm{Re}\, x > 0$$

Series representation

$$\beta(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k} \qquad [-x \notin \mathbb{N}]$$

$$\beta(x) = \sum_{k=0}^{\infty} \frac{1}{(x+2k)(x+2k+1)} \qquad [-x \notin \mathbb{N}]$$

B 函数定义

◆ 定义 5.1. Beta 函数

B 函数 (Beta function) 的定义为

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t\ \ (\mathrm{Re}\,x>0,\ \mathrm{Re}\,y>0)$$

上式的右边称为第一类欧拉 (Euler) 积分

其它形式

$$B(x,y)=2\int_0^{\frac{\pi}{2}}\sin^{2x-1}\theta\cos^{2y-1}\theta\mathrm{d}\theta\ \ (\mathrm{Re}\,x>0,\ \mathrm{Re}\,y>0)$$

$$B(x,y)=\int_0^1 \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\mathrm{d}t\ \ (\mathrm{Re}\,x>0,\ \mathrm{Re}\,y>0)$$

B 函数的其他形式

$$B(x,y) = 2\int_0^1 t^{2x-1}(1-t^2)^{y-1}\mathrm{d}t \quad (\operatorname{Re} x > 0,\ \operatorname{Re} y > 0)$$

$$B(x,y) = 2\int_0^{+\infty} \frac{t^{2x-1}}{(1+t^2)^{x+y}}\mathrm{d}t \quad (\operatorname{Re} x > 0,\ \operatorname{Re} y > 0)$$

$$B(x,y) = \int_1^{+\infty} \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\mathrm{d}t \quad (\operatorname{Re} x > 0,\ \operatorname{Re} y > 0)$$

$$B(x,y) = \int_0^{+\infty} \frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t = \int_0^{+\infty} \frac{t^{y-1}}{(1+t)^{x+y}}\mathrm{d}t$$

$$= \frac{1}{2}\int_0^{+\infty} \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\mathrm{d}t \quad (\operatorname{Re} x > 0,\ \operatorname{Re} y > 0)$$

Beta 函数的主要性质和公式

(1) 对称性: $\mathrm{B}(x,y)=\mathrm{B}(y,x)$

(2) 递推公式

(1) $\mathrm{B}(x+1,y)=\dfrac{x}{x+y}\mathrm{B}(x,y) \quad (\mathrm{Re}\,x>0,\mathrm{Re}\,y>0)$

(2) $\mathrm{B}(x,y+1)=\dfrac{y}{x+y}\mathrm{B}(x,y) \quad (\mathrm{Re}\,x>0,\mathrm{Re}\,y>0)$

(3) $\mathrm{B}(x+1,y+1)=\dfrac{xy}{(x+y+1)(x+y)}\mathrm{B}(x,y) \quad (\mathrm{Re}\,x>0,\ \mathrm{Re}\,y>0)$

(3) 如果 m,n 都是自然数, 则

$$\mathrm{B}(x,y)=\frac{(n-1)!(m-1)!}{(n+m-1)!}$$

(4) 余元公式: $\mathrm{B}(x,1-x)=\dfrac{\pi}{\sin x\pi} \ (0<x<1)$

(5) B 函数与 Γ 函数的关系:

$$\mathrm{B}(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \ (x>0,y>0)$$

Beta 函数的几个例题

例 1 计算积分 $\int_0^1 \frac{x^n dx}{1-x} \mathrm{d}x$

解

$$\int_0^1 \frac{x^n dx}{1-x} \mathrm{d}x = \mathrm{B}(n+1, 0) = \frac{\Gamma(n+1)\Gamma(0)}{\Gamma(n+1+0)} = 1$$

例 2 计算积分 $\int_0^1 \frac{(1-x)^n}{\sqrt{x}} \mathrm{d}x$

解

$$\int_0^1 \frac{(1-x)^n}{\sqrt{x}} \mathrm{d}x = \mathrm{B}(\tfrac{1}{2}, n+1) = \frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{\Gamma(n+1+\frac{1}{2})} = \frac{2^{n+1} n!}{(2n+1)!!}$$

Beta 函数的几个例题

例 1 计算积分 $\int_0^1 \frac{x^n dx}{1-x}\mathrm{d}x$

解

$$\int_0^1 \frac{x^n dx}{1-x}\mathrm{d}x = \mathrm{B}(n+1,0) = \frac{\Gamma(n+1)\Gamma(0)}{\Gamma(n+1+0)} = 1$$

例 2 计算积分 $\int_0^1 \frac{(1-x)^n}{\sqrt{x}}\mathrm{d}x$

解

$$\int_0^1 \frac{(1-x)^n}{\sqrt{x}}\mathrm{d}x = \mathrm{B}(\tfrac{1}{2},n+1) = \frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{\Gamma(n+1+\frac{1}{2})} = \frac{2^{n+1}n!}{(2n+1)!!}$$

Beta 函数的几个例题

例 1 计算积分 $\int_0^1 \frac{x^n dx}{1-x}\mathrm{d}x$

解

$$\int_0^1 \frac{x^n dx}{1-x}\mathrm{d}x = \mathrm{B}(n+1,0) = \frac{\Gamma(n+1)\Gamma(0)}{\Gamma(n+1+0)} = 1$$

例 2 计算积分 $\int_0^1 \frac{(1-x)^n}{\sqrt{x}}\mathrm{d}x$

解

$$\int_0^1 \frac{(1-x)^n}{\sqrt{x}}\mathrm{d}x = \mathrm{B}(\tfrac{1}{2},n+1) = \frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{\Gamma(n+1+\frac{1}{2})} = \frac{2^{n+1}n!}{(2n+1)!!}$$

Beta 函数的几个例题

例 3 计算积分 $\int_0^{+\infty} \frac{1}{(1+x^6)^2} \mathrm{d}x$

解

$$\int_0^{+\infty} \frac{1}{(1+x^6)^2} \mathrm{d}x \xlongequal[\mathrm{d}x=\frac{1}{3}\tan^{-\frac{2}{3}}t\mathrm{d}t]{x^3=\tan t} \int_0^{\frac{\pi}{2}} \frac{1}{\sec^2 x} \times \frac{1}{3}\tan^{-\frac{2}{3}} t\mathrm{d}t$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^{-\frac{2}{3}} t \cos^{\frac{8}{3}} t\mathrm{d}t$$

$$= \frac{1}{6} B\left(\frac{1}{6}, \frac{11}{6}\right)$$

$$= \frac{1}{6} \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{11}{6}\right)}{\Gamma(2)}$$

$$= \frac{5}{36}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right) = \frac{5\pi}{18}$$

Beta 函数的几个例题

例 3 计算积分 $\displaystyle\int_0^{+\infty}\frac{1}{(1+x^6)^2}\mathrm{d}x$

解

$$\begin{aligned}
\int_0^{+\infty}\frac{1}{(1+x^6)^2}\mathrm{d}x &\xlongequal[\mathrm{d}x=\frac{1}{3}\tan^{-\frac{2}{3}}t\mathrm{d}t]{x^3=\tan t}\int_0^{\frac{\pi}{2}}\frac{1}{\sec^2 x}\times\frac{1}{3}\tan^{-\frac{2}{3}}t\mathrm{d}t\\
&=\frac{1}{3}\int_0^{\frac{\pi}{2}}\sin^{-\frac{2}{3}}t\cos^{\frac{8}{3}}t\mathrm{d}t\\
&=\frac{1}{6}B\left(\frac{1}{6},\frac{11}{6}\right)\\
&=\frac{1}{6}\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{11}{6}\right)}{\Gamma(2)}\\
&=\frac{5}{36}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)=\frac{5\pi}{18}
\end{aligned}$$

Beta 函数的几个例题

例 4 计算积分 $\int_0^1 \frac{\mathrm{d}x}{\sqrt[n]{1-x^n}}$

解

$$I = \int_0^1 \frac{\mathrm{d}x}{\sqrt[n]{1-x^n}} \xlongequal{t=x^n} \frac{1}{n}\int_0^1 t^{\frac{1-n}{n}}(1-t)^{-\frac{1}{n}}\mathrm{d}t$$

$$= \frac{1}{n}\mathrm{B}\left(\tfrac{1}{n}, 1-\tfrac{1}{n}\right) = \frac{1}{n}\frac{\Gamma(\frac{1}{n})\Gamma(1-\frac{1}{n})}{\Gamma(1)}$$

$$\xlongequal{\text{余元公式}} \frac{\pi}{n\sin\frac{\pi}{n}}$$

注: 余元公式: 对于 $0 < z < 1$ 有

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$$

Beta 函数的几个例题

例 4 计算积分 $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt[n]{1-x^n}}$

解

$$\begin{aligned} I &= \int_0^1 \frac{\mathrm{d}x}{\sqrt[n]{1-x^n}} \\ &\xlongequal{t=x^n} \frac{1}{n}\int_0^1 t^{\frac{1-n}{n}}(1-t)^{-\frac{1}{n}}\mathrm{d}t \\ &= \frac{1}{n}\mathrm{B}(\tfrac{1}{n}, 1-\tfrac{1}{n}) = \frac{1}{n}\frac{\Gamma(\frac{1}{n})\Gamma(1-\frac{1}{n})}{\Gamma(1)} \\ &\xlongequal{\text{余元公式}} \frac{\pi}{n\sin\frac{\pi}{n}} \end{aligned}$$

注: 余元公式: 对于 $0 < z < 1$ 有

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

Beta 函数的几个例题

例 5 计算积分 $\int_0^{\frac{\pi}{2}} \sin^{\frac{5}{2}} x \mathrm{d}x$

解

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^{\frac{5}{2}} x \mathrm{d}x &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{2\times\frac{7}{4}-1} x \cos^{2\times\frac{1}{2}-1} x \mathrm{d}x \\ &= \frac{1}{2} B\left(\frac{7}{4}, \frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{4}\right)} \\ &= \frac{6\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{5\Gamma\left(\frac{1}{4}\right)} \approx 0.718884\end{aligned}$$

注: 用到的公式

$$\mathrm{B}(x, y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta \cos^{2y-1}\theta \mathrm{d}\theta \quad (\mathrm{Re}\, x > 0,\ \mathrm{Re}\, y > 0)$$

Beta 函数的几个例题

例 5 **计算积分** $\int_0^{\frac{\pi}{2}} \sin^{\frac{5}{2}} x\mathrm{d}x$

解

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^{\frac{5}{2}} x\mathrm{d}x &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{2\times\frac{7}{4}-1} x\cos^{2\times\frac{1}{2}-1} x\mathrm{d}x \\ &= \frac{1}{2}B\left(\frac{7}{4},\frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{4}\right)} \\ &= \frac{6\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{5\Gamma\left(\frac{1}{4}\right)} \approx 0.718884\end{aligned}$$

注: 用到的公式

$$\mathrm{B}(x,y) = 2\int_0^{\frac{\pi}{2}} \sin^{2x-1}\theta\cos^{2y-1}\theta\mathrm{d}\theta \ \ (\mathrm{Re}\, x > 0,\ \mathrm{Re}\, y > 0)$$

Beta 函数的几个例题

例 6 计算积分 $\displaystyle\int_0^1 \frac{5x^4(1+x^{10075})}{(1+x^5)^{2017}}\mathrm{d}x$

解 因为

$$\mathrm{B}(x,y)=\int_0^1 \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\mathrm{d}t \quad (\mathrm{Re}\ x>0, \mathrm{Re}\ y>0)$$

故

$$\begin{aligned}\int_0^1 \frac{5x^4(1+x^{10075})}{(1+x^5)^{2017}}\mathrm{d}x &\xlongequal{x^5=t} \int_0^1 \frac{1+t^{2015}}{(1+t)^{2017}}\mathrm{d}t \\ &= \int_0^1 \frac{x^{1-1}+t^{2016-1}}{(1+t)^{2017}}\mathrm{d}t \\ &= \mathrm{B}(1,2016) \\ &= \frac{0!2015!}{2016!} = \frac{1}{2016}\end{aligned}$$

Beta 函数的几个例题

例 6 计算积分 $\int_0^1 \frac{5x^4(1+x^{10075})}{(1+x^5)^{2017}}\mathrm{d}x$

解 因为

$$\mathrm{B}(x,y)=\int_0^1 \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\mathrm{d}t \quad (\mathrm{Re}\ x>0, \mathrm{Re}\ y>0)$$

故

$$\begin{aligned}\int_0^1 \frac{5x^4(1+x^{10075})}{(1+x^5)^{2017}}\mathrm{d}x &\xlongequal{x^5=t} \int_0^1 \frac{1+t^{2015}}{(1+t)^{2017}}\mathrm{d}t \\ &= \int_0^1 \frac{x^{1-1}+t^{2016-1}}{(1+t)^{2017}}\mathrm{d}t \\ &= \mathrm{B}(1,2016) \\ &= \frac{0!2015!}{2016!}=\frac{1}{2016}\end{aligned}$$

Beta 函数的几个例题

例 7 计算积分 $\int_0^{+\infty} \frac{x^{p-1}\ln x}{1+x}\mathrm{d}x \quad (0<p<1)$

解 注意到

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{x^{p-1}}{1+x}\right) = \frac{x^{p-1}\ln x}{1+x}$$

故可得

$$\begin{aligned}\int_0^{+\infty} \frac{x^{p-1}\ln x}{1+x}\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}p}\int_0^{+\infty}\frac{x^{p-1}\mathrm{d}x}{1+x} \\ &= \frac{\mathrm{d}}{\mathrm{d}p}\mathrm{B}(p,1-p) = \frac{\mathrm{d}}{\mathrm{d}p}(\Gamma(p)\Gamma(1-p)) \\ &= \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\pi}{\sin(p\pi)}\right) \\ &= -\frac{\pi^2\cos(p\pi)}{\sin^2(p\pi)}\end{aligned}$$

Beta 函数的几个例题

例 7 计算积分 $\int_0^{+\infty} \frac{x^{p-1}\ln x}{1+x}\mathrm{d}x \quad (0<p<1)$

解 注意到

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{x^{p-1}}{1+x}\right) = \frac{x^{p-1}\ln x}{1+x}$$

故可得

$$\begin{aligned}
\int_0^{+\infty} \frac{x^{p-1}\ln x}{1+x}\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}p}\int_0^{+\infty} \frac{x^{p-1}\mathrm{d}x}{1+x} \\
&= \frac{\mathrm{d}}{\mathrm{d}p}\mathrm{B}(p,1-p) = \frac{\mathrm{d}}{\mathrm{d}p}(\Gamma(p)\Gamma(1-p)) \\
&= \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\pi}{\sin(p\pi)}\right) \\
&= -\frac{\pi^2\cos(p\pi)}{\sin^2(p\pi)}
\end{aligned}$$

Beta 函数的几个例题

例 8 计算积分 $\displaystyle\int_0^\pi \frac{\mathrm{d}x}{\sqrt{3+\cos x}}$

解

$$\begin{aligned}
\int_0^\pi \frac{\mathrm{d}x}{\sqrt{3+\cos x}} &= \int_0^\pi \frac{\mathrm{d}x}{\sqrt{2+2\cos^2\frac{x}{2}}} \\
&\overset{u=\cos^2\frac{x}{2}}{=\!=\!=} \frac{1}{\sqrt{2}}\int_0^1 (1-u^2)^{-\frac{1}{2}}u^{-\frac{1}{2}}\mathrm{d}u \\
&\overset{t=u^2}{=\!=} \frac{1}{2\sqrt{2}}\int_0^1 (1-t)^{\frac{1}{2}-1}t^{\frac{1}{4}-1}\mathrm{d}t \\
&= \frac{1}{2\sqrt{2}}\mathrm{B}(\tfrac{1}{2},\tfrac{1}{4}) \\
&= \frac{1}{2\sqrt{2}}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})} = \frac{\sqrt{\pi}\Gamma^2(\frac{1}{4})}{2\sqrt{2}\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})} \\
&\overset{\text{余元公式}}{=\!=\!=} \frac{1}{4\sqrt{\pi}}\Gamma^2(\tfrac{1}{4}) \approx 1.85407
\end{aligned}$$

Beta 函数的几个例题

例 8 计算积分 $\displaystyle\int_0^{\pi}\frac{\mathrm{d}x}{\sqrt{3+\cos x}}$

解

$$\begin{aligned}
\int_0^{\pi}\frac{\mathrm{d}x}{\sqrt{3+\cos x}} &= \int_0^{\pi}\frac{\mathrm{d}x}{\sqrt{2+2\cos^2\frac{x}{2}}} \\
&\xlongequal{u=\cos^2\frac{x}{2}} \frac{1}{\sqrt{2}}\int_0^1(1-u^2)^{-\frac{1}{2}}u^{-\frac{1}{2}}\mathrm{d}u \\
&\xlongequal{t=u^2} \frac{1}{2\sqrt{2}}\int_0^1(1-t)^{\frac{1}{2}-1}t^{\frac{1}{4}-1}\mathrm{d}t \\
&= \frac{1}{2\sqrt{2}}\mathrm{B}\left(\tfrac{1}{2},\tfrac{1}{4}\right) \\
&= \frac{1}{2\sqrt{2}}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} \\
&\xlongequal{\text{余元公式}} \frac{1}{4\sqrt{\pi}}\Gamma^2\left(\tfrac{1}{4}\right) \approx 1.85407
\end{aligned}$$

Beta 函数的几个例题

例 9 计算积分 $\displaystyle\int_0^{+\infty}\frac{1}{1+x^n}\mathrm{d}x,\ n>1$

解

$$
\begin{aligned}
I &= \int_0^{+\infty}\frac{1}{1+x^n}\mathrm{d}x \\
&\xlongequal{x=\sqrt[n]{\tan^2\theta}} \frac{2}{n}\int_0^{\frac{\pi}{2}}\cos^{1-\frac{2}{n}}\theta\sin^{\frac{2}{n}-1}\theta\mathrm{d}\theta \\
&= \frac{1}{n}\mathrm{B}\left(1-\frac{1}{n},\frac{1}{n}\right) \\
&= \frac{1}{n}\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right) \\
&\xlongequal{\text{余元公式}} \frac{\pi}{n\sin\frac{\pi}{n}}
\end{aligned}
$$

余元公式: 对于 $0<z<1$ 有 $\Gamma(z)\Gamma(1-z)=\dfrac{\pi}{\sin\pi z}$

Beta 函数的几个例题

例 9 计算积分 $\int_0^{+\infty} \frac{1}{1+x^n} \mathrm{d}x, n > 1$

解

$$
\begin{aligned}
I &= \int_0^{+\infty} \frac{1}{1+x^n} \mathrm{d}x \\
&\xlongequal{x=\sqrt[n]{\tan^2\theta}} \frac{2}{n} \int_0^{\frac{\pi}{2}} \cos^{1-\frac{2}{n}} \theta \sin^{\frac{2}{n}-1} \theta \mathrm{d}\theta \\
&= \frac{1}{n} \mathrm{B}\left(1-\frac{1}{n}, \frac{1}{n}\right) \\
&= \frac{1}{n} \Gamma\left(1-\frac{1}{n}\right) \Gamma\left(\frac{1}{n}\right) \\
&\xlongequal{\text{余元公式}} \frac{\pi}{n \sin \frac{\pi}{n}}
\end{aligned}
$$

余元公式: 对于 $0 < z < 1$ 有 $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$

Beta 函数的几个例题

例 10 计算积分 $\displaystyle\int_0^1 \frac{x^{a-1}(1-x)^{b-1}}{(x+p)^{a+b}}\mathrm{d}x,\ (a,b,p>0)$

解 作变量替换, 令 $y=(1+p)\dfrac{x}{x+p}$

则

$$\mathrm{d}y=\frac{p(p+1)}{(x+p)^2}\mathrm{d}x \Rightarrow \mathrm{d}x=\frac{(x+p)^2}{p(p+1)}\mathrm{d}y$$

且注意到

$$1-y=1-(1+p)\frac{x}{x+p}=\frac{p(1-x)}{x+p}$$

故有

$$1-x=\frac{x+p}{p}(1-y)$$

Beta 函数的几个例题

例 10 计算积分 $\displaystyle\int_0^1 \frac{x^{a-1}(1-x)^{b-1}}{(x+p)^{a+b}}\mathrm{d}x, (a,b,p>0)$

解 作变量替换, 令 $y=(1+p)\dfrac{x}{x+p}$

则

$$\mathrm{d}y=\frac{p(p+1)}{(x+p)^2}\mathrm{d}x \Rightarrow \mathrm{d}x=\frac{(x+p)^2}{p(p+1)}\mathrm{d}y$$

且注意到

$$1-y=1-(1+p)\frac{x}{x+p}=\frac{p(1-x)}{x+p}$$

故有

$$1-x=\frac{x+p}{p}(1-y)$$

Beta 函数的几个例题

$$\begin{aligned}
I &= \int_0^1 \frac{x^{a-1}(1-x)^{b-1}}{(x+p)^{a+b}}\mathrm{d}x \\
&\xlongequal{y=(1+p)\frac{x}{x+p}} \int_0^1 \frac{x^{a-1}(\frac{x+p}{p}(1-y))^{b-1}}{(x+p)^{a+b}}\frac{(x+p)^2}{p(p+1)}\mathrm{d}y \\
&= \frac{1}{(1+p)p^b}\int_0^1 \frac{x^{a-1}(1-y)^{b-1}}{(x+p)^{a-1}}\mathrm{d}y \\
&= \frac{1}{(1+p)^a p^b}\int_0^1 y^{a-1}(1-y)^{b-1}\mathrm{d}y \\
&= \frac{1}{(1+p)^a p^b}\mathrm{B}(a,b)
\end{aligned}$$

Beta function

$$\mathrm{B}(x,y) = \int_0^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t \quad p,q>0$$

Beta 函数的几个例题

例 11 计算积分 $\int_{-1}^{1}\frac{(1+x)^{2m-1}(1-x)^{2n-1}}{(1+x^2)^{m+n}}\mathrm{d}x,\ (m,n>0)$

解 令 $t=\frac{(1+x)^2}{2(1+x^2)}$，则 $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(1+x)^2}{2(1+x^2)}\right)=\frac{1-x^2}{(1+x^2)^2}\mathrm{d}t$

$$1-t=\frac{(1-x)^2}{2(1+x^2)}\Rightarrow(1-x)^2=2(1-t)(1+x^2)$$

故

$$\begin{aligned}I&=\int_{-1}^{1}\frac{(1+x)^{2m-1}(1-x)^{2n-1}}{(1+x^2)^{m+n}}\mathrm{d}x\\&=2\int_{0}^{1}\frac{(1+x)^{2m-1}(1-x)^{2n-1}}{(1+x^2)^{m+n}}\mathrm{d}x\\&\xlongequal{u=\frac{1}{2}\frac{(1+x)^2}{(1+x^2)}}2^{m+n-2}\int_0^1 t^{m-1}(1-t)^{n-1}\mathrm{d}t\\&=2^{m+n-2}\mathrm{B}(m,n)\end{aligned}$$

Beta 函数的几个例题

例 11 计算积分 $\displaystyle\int_{-1}^{1}\frac{(1+x)^{2m-1}(1-x)^{2n-1}}{(1+x^2)^{m+n}}\mathrm{d}x,\ (m,n>0)$

解 令 $t=\dfrac{(1+x)^2}{2(1+x^2)}$, 则 $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{(1+x)^2}{2(1+x^2)}\right)=\dfrac{1-x^2}{(1+x^2)^2}\mathrm{d}t$

$$1-t=\frac{(1-x)^2}{2(1+x^2)}\Rightarrow(1-x)^2=2(1-t)(1+x^2)$$

故

$$\begin{aligned}
I&=\int_{-1}^{1}\frac{(1+x)^{2m-1}(1-x)^{2n-1}}{(1+x^2)^{m+n}}\mathrm{d}x\\
&=2\int_{0}^{1}\frac{(1+x)^{2m-1}(1-x)^{2n-1}}{(1+x^2)^{m+n}}\mathrm{d}x\\
&\xlongequal{u=\frac{1}{2}\frac{(1+x)^2}{(1+x^2)}}2^{m+n-2}\int_0^1 t^{m-1}(1-t)^{n-1}\mathrm{d}t\\
&=2^{m+n-2}\mathrm{B}(m,n)
\end{aligned}$$

Beta 函数的几个例题

例 12 计算积分 $\int_0^{\pi}\frac{\sin^n x}{(1+k\cos x)^n}\mathrm{d}x,\ (0<|k|<1)$

解

$$I=\int_0^{\pi}\frac{\sin^n x}{(1+k\cos x)^n}\mathrm{d}x$$

$$\xlongequal{t=\tan(\frac{x}{2})}\frac{2^n}{(1+k)^n}\int_0^{+\infty}\frac{t^{n-1}}{(1+\alpha^2t^2)^n}\mathrm{d}t,\ \alpha=\sqrt{\frac{1-k}{1+k}}$$

$$\xlongequal{\alpha t=\sqrt{s}}\frac{2^n}{(1+k)^n}\cdot\frac{1}{2\alpha^n}\int_0^{+\infty}\frac{s^{\frac{1}{2}n-1}}{(1+s)^n}\mathrm{d}s$$

$$=\frac{2^{n-1}}{(1-k^2)^{\frac{n}{2}}}\mathrm{B}\left(\frac{n}{2},\frac{n}{2}\right)$$

Beta function

$$\int_0^{+\infty}\frac{t^{p-1}}{(1+t)^{p+q}}\mathrm{d}t\quad p,q>0$$

Beta 函数的几个例题

例 12 计算积分 $\int_0^{\pi} \frac{\sin^n x}{(1+k\cos x)^n} dx, (0<|k|<1)$

解

$$
\begin{aligned}
I &= \int_0^{\pi} \frac{\sin^n x}{(1+k\cos x)^n} dx \\
&\xlongequal{t=\tan(\frac{x}{2})} \frac{2^n}{(1+k)^n} \int_0^{+\infty} \frac{t^{n-1}}{(1+\alpha^2 t^2)^n} dt, \alpha = \sqrt{\frac{1-k}{1+k}} \\
&\xlongequal{\alpha t=\sqrt{s}} \frac{2^n}{(1+k)^n} \cdot \frac{1}{2\alpha^n} \int_0^{+\infty} \frac{s^{\frac{1}{2}n-1}}{(1+s)^n} ds \\
&= \frac{2^{n-1}}{(1-k^2)^{\frac{n}{2}}} \mathrm{B}\left(\frac{n}{2}, \frac{n}{2}\right)
\end{aligned}
$$

Beta function

$$
\int_0^{+\infty} \frac{t^{p-1}}{(1+t)^{p+q}} dt \quad p,q>0
$$

Beta 函数的几个例题

例 13 证明 $\displaystyle\sum_{k=0}^{\infty} C_n^k \frac{(-1)^k}{m+k+1} = \frac{m!n!}{(m+n+1)!}$

证

$$\begin{aligned}
\sum_{k=0}^{\infty} C_n^k \frac{(-1)^k}{m+k+1} &= \sum_{k=0}^{\infty} C_n^k (-1)^k \int_0^1 x^{m+k} \mathrm{d}x \\
&= \int_0^1 \sum_{k=0}^{\infty} C_n^k (-1)^k x^{m+k} \mathrm{d}x \\
&= \int_0^1 x^m (1-x)^n \mathrm{d}x \\
&= \mathrm{B}(m+1, n+1) \\
&= \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+1)} \\
&= \frac{m!n!}{(m+n+1)!}
\end{aligned}$$

Beta 函数的几个例题

例 13 证明 $\sum_{k=0}^{\infty} C_n^k \frac{(-1)^k}{m+k+1} = \frac{m!n!}{(m+n+1)!}$

证

$$\begin{aligned}
\sum_{k=0}^{\infty} C_n^k \frac{(-1)^k}{m+k+1} &= \sum_{k=0}^{\infty} C_n^k (-1)^k \int_0^1 x^{m+k} \mathrm{d}x \\
&= \int_0^1 \sum_{k=0}^{\infty} C_n^k (-1)^k x^{m+k} \mathrm{d}x \\
&= \int_0^1 x^m (1-x)^n \mathrm{d}x \\
&= \mathrm{B}(m+1, n+1) \\
&= \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+1)} \\
&= \frac{m!n!}{(m+n+1)!}
\end{aligned}$$

Gamma 函数的性质公式证明

证明: $\alpha_1+\alpha_2+\cdots+\alpha_n=\beta_1+\beta_2+\cdots+\beta_n$ then

$$\frac{\Gamma(\beta_1)\cdot\cdots\cdot\Gamma(\beta_n)}{\Gamma(\alpha_1)\cdot\cdots\cdot\Gamma(\alpha_n)}=\prod_{k\geqslant 0}\frac{(k+\alpha_1)\cdot\cdots\cdot(k+\alpha_n)}{(k+\beta_1)\cdot\cdots\cdot(k+\beta_n)}$$

证

$$\frac{\Gamma(\beta_1)\cdot\cdots\cdot\Gamma(\beta_n)}{\Gamma(\alpha_1)\cdot\cdots\cdot\Gamma(\alpha_n)}=\prod_{j=1}^{n}\frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)}=\lim_{m\to\infty}\prod_{j=1}^{n}\frac{\frac{m^{\beta_j}m!}{\beta_j(\beta_j+1)\cdot\cdots\cdot(\beta_j+m)}}{\frac{m^{\alpha_j}m!}{\alpha_j(\alpha_j+1)\cdot\cdots\cdot(\alpha_j+m)}}$$

$$=\lim_{m\to\infty}\prod_{j=1}^{n}m^{\beta_j-\alpha_j}\prod_{k=0}^{m}\frac{\alpha_j+k}{\beta_j+k}$$

$$=\lim_{m\to\infty}\prod_{k=0}^{m}\prod_{j=1}^{n}\frac{\alpha_j+k}{\beta_j+k}$$

$$=\lim_{m\to\infty}\prod_{k=0}^{m}\frac{(k+\alpha_1)\cdot\cdots\cdot(k+\alpha_n)}{(k+\beta_1)\cdot\cdots\cdot(k+\beta_n)}$$

Gamma 函数的性质公式证明

证明: $\alpha_1+\alpha_2+\cdots+\alpha_n=\beta_1+\beta_2+\cdots+\beta_n$ then

$$\frac{\Gamma(\beta_1)\cdot\cdots\cdot\Gamma(\beta_n)}{\Gamma(\alpha_1)\cdot\cdots\cdot\Gamma(\alpha_n)}=\prod_{k\geqslant 0}\frac{(k+\alpha_1)\cdot\cdots\cdot(k+\alpha_n)}{(k+\beta_1)\cdot\cdots\cdot(k+\beta_n)}$$

证

$$\frac{\Gamma(\beta_1)\cdot\cdots\cdot\Gamma(\beta_n)}{\Gamma(\alpha_1)\cdot\cdots\cdot\Gamma(\alpha_n)}=\prod_{j=1}^{n}\frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)}=\lim_{m\to\infty}\prod_{j=1}^{n}\frac{\frac{m^{\beta_j}m!}{\beta_j(\beta_j+1)\cdot\cdots\cdot(\beta_j+m)}}{\frac{m^{\alpha_j}m!}{\alpha_j(\alpha_j+1)\cdot\cdots\cdot(\alpha_j+m)}}$$

$$=\lim_{m\to\infty}\prod_{j=1}^{n}m^{\beta_j-\alpha_j}\prod_{k=0}^{m}\frac{\alpha_j+k}{\beta_j+k}$$

$$=\lim_{m\to\infty}\prod_{k=0}^{m}\prod_{j=1}^{n}\frac{\alpha_j+k}{\beta_j+k}$$

$$=\lim_{m\to\infty}\prod_{k=0}^{m}\frac{(k+\alpha_1)\cdot\cdots\cdot(k+\alpha_n)}{(k+\beta_1)\cdot\cdots\cdot(k+\beta_n)}$$

Legendre 加倍公式

证明: Legendre 加倍公式：

$$\sqrt{\pi}\Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right), s > 0$$

证: 记 $I(s) = \int_0^1 \frac{\mathrm{d}x}{(1+x^{\frac{1}{s}})^{2s}}$. 令 $x = \tan^{2s} t$,

则 $\mathrm{d}x = \sin^{2s-1} t \cos^{-2s-1} t\mathrm{d}t$, $(1+x^{\frac{1}{s}})^{2s} = \sec^{4s} t$,

从而

$$I(s) = \int_0^1 \frac{\mathrm{d}x}{(1+x^{\frac{1}{s}})^{2s}} = 2s\int_0^{\frac{\pi}{4}} (\sin t\cos t)^{2s-1}\mathrm{d}t$$

$$= s2^{1-2s}\int_0^{\frac{\pi}{2}} \sin^{2s-1} u\mathrm{d}u = 2^{-2s}sB(\frac{1}{2}, s)$$

$$= 2^{-2s}s\frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(\frac{1}{2}+s)} = 2^{-2s}\sqrt{\pi}s\frac{\Gamma(s)}{\Gamma(\frac{1}{2}+s)}.$$

Legendre 加倍公式

证明: Legendre 加倍公式:

$$\sqrt{\pi}\Gamma(2s) = 2^{2s-1}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right), s>0$$

证: 记 $I(s) = \int_0^1 \frac{\mathrm{d}x}{(1+x^{\frac{1}{s}})^{2s}}$. 令 $x = \tan^{2s} t$,

则 $\mathrm{d}x = \sin^{2s-1} t \cos^{-2s-1} t\mathrm{d}t$, $(1+x^{\frac{1}{s}})^{2s} = \sec^{4s} t$,

从而

$$I(s) = \int_0^1 \frac{\mathrm{d}x}{(1+x^{\frac{1}{s}})^{2s}} = 2s\int_0^{\frac{\pi}{4}} (\sin t\cos t)^{2s-1}\mathrm{d}t$$

$$= s2^{1-2s}\int_0^{\frac{\pi}{2}} \sin^{2s-1} u\mathrm{d}u = 2^{-2s}sB(\frac{1}{2}, s)$$

$$= 2^{-2s}s\frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(\frac{1}{2}+s)} = 2^{-2s}\sqrt{\pi}s\frac{\Gamma(s)}{\Gamma(\frac{1}{2}+s)}.$$

Legendre 加倍公式

另一方面

$$I(s)=\int_0^1\frac{\mathrm{d}x}{(1+x^{\frac{1}{s}})^{2s}}=\int_1^{+\infty}\frac{\mathrm{d}x}{(1+x^{\frac{1}{s}})^{2s}},$$

从而

$$I(s)=\frac{1}{2}\int_0^{+\infty}\frac{\mathrm{d}x}{(1+x^{\frac{1}{s}})^{2s}}=s\int_0^{\frac{\pi}{2}}(\sin t\cos t)^{2s-1}\mathrm{d}t=\frac{sB(s,s)}{2}=\frac{s\Gamma^2(s)}{2\Gamma(2s)}.$$

因此

$$2^{-2s}\sqrt{\pi}s\frac{\Gamma(s)}{\Gamma(\frac{1}{2}+s)}=\frac{s\Gamma^2(s)}{2\Gamma(2s)}.$$

从而

$$\sqrt{\pi}\Gamma(2s)=2^{2s-1}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right),s>0.$$

余元公式

证明: 对于 $0 < z < 1$ 有

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

证:

$$\begin{aligned}
\Gamma(z)\Gamma(1-z) &= \mathrm{B}(1-z, z)\Gamma(1-z+z) \\
&= \mathrm{B}(1-z, z)\Gamma(1) = \int_0^1 t^{-z}(1-t)^{z-1}\,\mathrm{d}t \\
&\xlongequal[1-t=\frac{x}{1+x}]{t=\frac{1}{1+x}} \int_{+\infty}^0 \left(\frac{1}{1+x}\right)^{-z}\left(\frac{x}{1+x}\right)^{z-1}\frac{-\mathrm{d}x}{(1+x)^2} \\
&= \int_0^{+\infty}\frac{x^{z-1}}{1+x}\,\mathrm{d}x = \frac{\pi}{\sin \pi z}
\end{aligned}$$

Beta 函数的性质公式证明

证明: $\mathrm{B}(x,y)=\int_0^{+\infty}\frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t=\int_0^1\frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\mathrm{d}t$ 其中 $x>0,y>0$

证　利用换元法，令 $t=\frac{u}{1+u}$，则 $\mathrm{d}t=\frac{1}{(1+u)^2}\mathrm{d}u$

$$\mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$$

$$\overset{t=\frac{u}{1+u}}{=\!=\!=\!=}\int_0^{\infty}\frac{u^{x-1}}{(1+u)^{x-1}}\frac{1}{(1+u)^{y-1}}\frac{1}{(1+u)^2}\mathrm{d}u$$

$$=\int_0^{+\infty}\frac{u^{x-1}}{(1+u)^{x+y}}\mathrm{d}u$$

$$=\int_0^{+\infty}\frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t$$

Beta 函数的性质公式证明

证明: $\mathrm{B}(x,y)=\int_0^{+\infty}\frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t=\int_0^1\frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\mathrm{d}t$ 其中 $x>0, y>0$

证 利用换元法, 令 $t=\frac{u}{1+u}$, 则 $\mathrm{d}t=\frac{1}{(1+u)^2}\mathrm{d}u$

$$\begin{aligned}
\mathrm{B}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t \\
&\overset{t=\frac{u}{1+u}}{=\!=\!=} \int_0^{\infty}\frac{u^{x-1}}{(1+u)^{x-1}}\frac{1}{(1+u)^{y-1}}\frac{1}{(1+u)^2}\mathrm{d}u \\
&= \int_0^{+\infty}\frac{u^{x-1}}{(1+u)^{x+y}}\mathrm{d}u \\
&= \int_0^{+\infty}\frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t
\end{aligned}$$

Beta 函数的性质公式证明

由此, 继续应用换元法, 有:

$$\begin{aligned}
\mathrm{B}(x,y) &= \int_0^{+\infty} \frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t \\
&= \int_0^{1} \frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t + \underbrace{\int_1^{+\infty} \frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t}_{t=\frac{1}{u}} \\
&= \int_0^{1} \frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t + \int_1^{0} \frac{u^{1-x}}{(1+u)^{x+y}u^{-x-y}}\left(-\frac{\mathrm{d}u}{u^2}\right) \\
&= \int_0^{1} \frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t + \int_0^{1} \frac{u^{y-1}}{(1+u)^{x+y}}\mathrm{d}u \\
&= \int_0^{1} \frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t + \int_0^{1} \frac{t^{y-1}}{(1+t)^{x+y}}\mathrm{d}t \\
&= \int_0^{1} \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\mathrm{d}t
\end{aligned}$$

Beta 函数的性质公式证明

证明: $\mathrm{B}(p,q)=\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\quad (p>0,q>0)$

证 因为

$$\Gamma(p)=\int_0^{+\infty} x^{p-1}e^{-x}\mathrm{d}x$$

$$\Gamma(q)=\int_0^{+\infty} x^{q-1}e^{-x}\mathrm{d}x$$

故

$$\begin{aligned}\Gamma(p)\Gamma(q)&=\int_0^{+\infty} x^{p-1}e^{-x}\mathrm{d}x\int_0^{+\infty} y^{q-1}e^{-y}\mathrm{d}y\\&=\int_0^{+\infty}\mathrm{d}y\int_0^{+\infty} x^{p-1}y^{q-1}e^{-(x+y)}\mathrm{d}x\end{aligned}$$

做变量替换, 令 $x=uv, y=u(1-v)$

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做变量替换, 令 $x=uv,y=u(1-v)$

Beta 函数的性质公式证明

则其雅可比行列式 J 为

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix}$$

$$=-uv-u(1-v)=-u$$

$D\to D'$ 即:

$$\begin{cases} x=0 & \to u=0, v=0 \\ y=0 & \to u=0, v=1 \end{cases}$$

故

$$\Gamma(p)\Gamma(q)=\int_0^{+\infty}\int_0^1 (uv)^{p-1}\left[u(1-v)\right]^{q-1}e^{-u}u\mathrm{d}v\mathrm{d}u$$
$$=\int_0^{+\infty}u^{p+q-1}e^{-u}\mathrm{d}u\int_0^1 v^{p-1}(1-v)^{q-1}\mathrm{d}v$$

第一类椭圆积分

♦ 定义 7.1. 第一类不完全椭圆积分

$$F(k,\varphi)=\int_0^{\sin\varphi}\frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

$$=\int_0^{\varphi}\frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}\quad(k^2<1)$$

$$F(\phi,k)=\int_0^{\phi}\frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$=\int_0^{\sin\varphi}\frac{\mathrm{d}t}{\sqrt{(1-k^2t^2)(1-t^2)}}$$

$$F(\phi|m)=\int_0^{\phi}\frac{\mathrm{d}\theta}{\sqrt{1-m\sin^2\theta}}$$

$$=\int_0^{\sin\varphi}\frac{\mathrm{d}t}{\sqrt{(1-mt^2)(1-t^2)}}$$

第一类椭圆积分

♦ 定义 7.2. 第一类完全椭圆积分

$$K = K(k) = K\left(k, \frac{\pi}{2}\right) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}$$

其中 $k^2 < 1$

第二类椭圆积分

♦ 定义 7.3. 第二类不完全椭圆积分

$$E(k,\varphi) = \int_0^{\sin\varphi} \sqrt{\frac{1-k^2t^2}{1-t^2}}\mathrm{d}t = \int_0^{\varphi} \sqrt{1-k^2\sin^2\theta}\mathrm{d}\theta$$

$$E\left(\varphi\middle|m\right) = \int_0^{\sin\varphi} \sqrt{\frac{1-mt^2}{1-t^2}}\mathrm{d}t = \int_0^{\phi} \sqrt{1-m\sin^2\theta}\mathrm{d}\theta$$

其中 $k^2 < 1$

第二类椭圆积分

♦ 定义 7.4. 第二类完全椭圆积分

$$
\begin{aligned}
E = E(k) = E\left(k, \frac{\pi}{2}\right) &= \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\mathrm{d}t \\
&= \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\mathrm{d}\theta
\end{aligned}
$$

其中 $k^2 < 1$

第三类椭圆积分

♦ 定义 7.5. 第三类不完全椭圆积分

$$\Pi(h,k,\varphi) = \int_0^{\sin\varphi} \frac{dt}{(1+ht^2)\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\varphi} \frac{d\theta}{(1+h\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}$$

$$\Pi(n;\phi,k) = \int_0^{\sin\phi} \frac{dt}{(1-nt^2)\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\phi} \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}$$

$$\Pi\left(n;\phi\middle|m\right) = \int_0^{\sin\varphi} \frac{dt}{(1-nt^2)\sqrt{(1-t^2)(1-mt^2)}} = \int_0^{\varphi} \frac{d\theta}{(1-n\sin^2\theta)\sqrt{1-m\sin^2\theta}}$$

第三类椭圆积分

♦ 定义 7.6. 第三类完全椭圆积分

$$\begin{aligned}\Pi(h,k) &= \Pi\left(h,k,\frac{\pi}{2}\right)\\ &= \int_0^1 \frac{\mathrm{d}t}{(1+ht^2)\sqrt{(1-t^2)(1-k^2t^2)}}\\ &= \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{(1+h\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}\end{aligned}$$

其中 $k^2<1, h$ 为非负整数

第二类不完全椭圆积分

例 1 求不定积分 $\int \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\, \mathrm{d}\theta$

解

$$
\begin{aligned}
&\int \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\, \mathrm{d}\theta \\
&= \int \sqrt{b^2 + (a^2 - b^2)\sin^2\theta}\, \mathrm{d}\theta \\
&= b \int \sqrt{1 - \frac{b^2 - a^2}{b^2}\sin^2\theta}\, \mathrm{d}\theta \\
&= bE\left(x \middle| \frac{b^2 - a^2}{b^2}\right) + C \\
&= bE\left(x \middle| 1 - \frac{a^2}{b^2}\right) + C
\end{aligned}
$$

第二类不完全椭圆积分

例 1 求不定积分 $\int \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\,\mathrm{d}\theta$

解

$$\begin{aligned}
&\int \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}\,\mathrm{d}\theta \\
&= \int \sqrt{b^2 + (a^2 - b^2)\sin^2\theta}\,\mathrm{d}\theta \\
&= b\int \sqrt{1 - \frac{b^2 - a^2}{b^2}\sin^2\theta}\,\mathrm{d}\theta \\
&= bE\left(x \middle| \frac{b^2 - a^2}{b^2}\right) + C \\
&= bE\left(x \middle| 1 - \frac{a^2}{b^2}\right) + C
\end{aligned}$$

第二类不完全椭圆积分

例 2 计算下面两个积分的比值：

$$\int_0^1 \frac{1}{\sqrt{1+t^4}}\mathrm{d}t\ ,\quad \int_0^1 \frac{1}{\sqrt{1-t^4}}\mathrm{d}t$$

解 由

$$\int_0^1 \frac{1}{\sqrt{1+t^4}}\mathrm{d}t = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1+t^2)^2-2t^2}}$$

$$= \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-\frac{1}{2}\left(\frac{2t}{1+t^2}\right)^2}}\frac{2}{1+t^2}\mathrm{d}t$$

$$\left(t=\tan\frac{\theta}{2}\right) \quad = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \frac{1}{2}\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)$$

第二类不完全椭圆积分

例 2 计算下面两个积分的比值：

$$\int_0^1 \frac{1}{\sqrt{1+t^4}}\mathrm{d}t\,,\quad \int_0^1 \frac{1}{\sqrt{1-t^4}}\mathrm{d}t$$

解 由

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1+t^4}}\mathrm{d}t &= \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1+t^2)^2-2t^2}}\\ &= \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-\frac{1}{2}\left(\frac{2t}{1+t^2}\right)^2}}\frac{2}{1+t^2}\mathrm{d}t\\ \left(t=\tan\frac{\theta}{2}\right)\quad &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \frac{1}{2}\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)\end{aligned}$$

第二类不完全椭圆积分

和

$$\int_0^1 \frac{1}{\sqrt{1-t^4}}\mathrm{d}t = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1+\cos^2\theta}} \quad (t=\cos\theta)$$
$$= \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{2-\sin^2\theta}}$$
$$= \frac{1}{\sqrt{2}}\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)$$

我们可以得到

$$\boxed{\frac{\int_0^1 \frac{1}{\sqrt{1-x^4}}\mathrm{d}x}{\int_0^1 \frac{1}{\sqrt{1+x^4}}\mathrm{d}x} = \frac{\frac{1}{\sqrt{2}}\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)}{\frac{1}{2}\mathbf{K}\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{2}}$$

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致谢

欢迎老师批评指正!

Thank you!

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